Landau Damping Simulation Models

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Content

Introduction

Landau damping[1] is one of the most interesting phenomena found in plasma physics. However, the mathematical derivation and physical understanding of it are usually headache, especially for beginners.

Here, I will tell how to use simple and short codes to study this phenomena. A shortest code to produce Landau damping accurately can be even **less than 10** lines!

[1] I think I can safely say that nobody understands Landau damping fully.

Linear simulation model: equation

We focus on the electrostatic 1D (ES1D) Vlasov-Poisson system (ion immobile).

The simplest method to study Landau damping is solving the following equations

$$\partial_t \delta f = -ikv\delta f + \delta E \partial_v f_0, \tag{1a}$$

$$ik\delta E = -\int \delta f dv, \tag{1b}$$

Example code

```
k=0.4; dt=0.01; nt=8000; dv=0.1; vv=-8:dv:8;
df0dv=-vv.*exp(-vv.^2./2)/sqrt(2*pi);
df=0.*vv+0.1.*exp(-(vv-2.0).^2); tt=linspace(0,nt*
    dt,nt+1);
dE=zeros(1,nt+1); dE(1)=0.01;
for it=1:nt
    df=df+dt.*(-1i*k.*vv.*df+dE(it).*df0dv);
    dE(it+1)=(1i/k)*sum(df)*dv;
end
plot(tt,real(dE)); xlabel('t'); ylabel('Re(dE)');
```

Simulation result

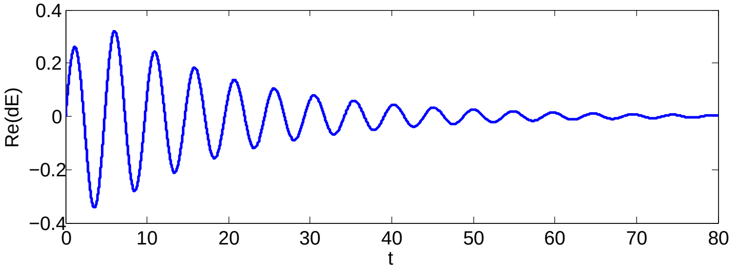

Figure 1: Linear simulation of Landau damping.

Exercise

Exercise 1: Solving the following fluid equations

$$\begin{aligned} \partial_t \delta n &= -ik\delta u, && (2a) \\ \partial_t \delta u &= -\delta E - 3ik\delta n, && (2b) \\ ik\delta E &= -\delta n, && (2c) \end{aligned}$$

using the above method to reproduce the Langmuir wave

$$\omega^2 = 1 + 3k^2. \tag{3}$$

Dispersion relation

$$D(k,\omega) = 1 - \frac{1}{k^2}\int_C \frac{\partial f_0/\partial v}{v-\omega/k}dv = 0, \tag{4}$$

where C is the Landau integral contour. For Maxwellian distribution $f_0 = \frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}$, we will meet the well-known plasma dispersion function (PDF)

$$Z_M(\zeta) = \frac{1}{\sqrt{\pi}}\int_C \frac{e^{-z^2}}{z-\zeta}dz. \tag{5}$$

Hence, (4) is rewritten to

$$D(k,\omega) = 1 - \frac{1}{k^2}\frac{1}{2}Z'_M(\zeta) = 0. \tag{6}$$

Numerical solutions

Table 1: Numerical solutions of the Landau damping dispersion relation

$k\lambda_D$	ω_r/ω_{pe}	γ_r/ω_{pe}
0.1	1.0152	-4.75613E-15
0.2	1.06398	-5.51074E-05
0.3	1.15985	-0.0126204
0.4	1.28506	-0.066128
0.5	1.41566	-0.153359
0.6	1.54571	-0.26411
0.7	1.67387	-0.392401
0.8	1.7999	-0.534552
0.9	1.92387	-0.688109
1.0	2.0459	-0.85133
1.5	2.63233	-1.77571
2	3.18914	-2.8272

Comparison of DR and linear simulation

Adding some diagnosis lines to the code. Perfect agreement: $\omega^{\text{theory}} = 1.28506 - 0.066128i$ and $\omega^{\text{simulation}} = 1.2849 - 0.06627i$

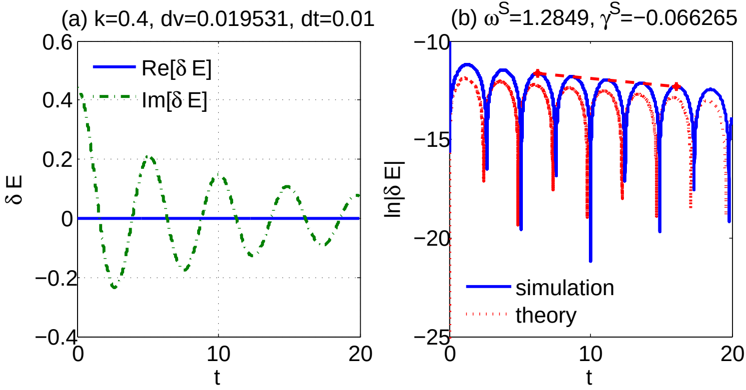

Figure 2: Linear simulation of Landau damping and compared with theory.

More

Note: Several related topics have been omitted here, e.g., Case-van Kampen ballistic modes (eigenmode problem), non-physical recurrence effect (the Poincaré recurrence) at $T_R = 2\pi/(k\Delta v)$[1], solving the dispersion relation with general equilibrium distribution functions (not limited to Maxwellian), advanced schemes (e.g., 4th R-K), and so on. One can refer Ref.[2] and references in for more details.

PIC simulation: equations

Normalized equations (Lagrangian approach)

$$d_t x_i = v_i, \tag{7a}$$

$$d_t v_i = -E(x_i), \tag{7b}$$

$$d_x E(x_j) = 1 - n(x_j), \tag{7c}$$

where $i = 1, 2, \cdots, N_p$ is particle (marker) label and $j = 0, 1, \cdots, N_g - 1$ is grid label. The particles i can be every where, whereas the field is discrete in grids $x_j = j\Delta x$. $\Delta x = L/N_g$. Domain $0 < x < L$ [note: $\int n(x)dx = L$], periodic boundary conditions $n(0) = n(L)$ and $\left\langle E(x) \right\rangle_x = 0$. Any particle crosses the right boundary of the solution domain must reappear at the left boundary with the same velocity, and vice versa.

The initial probability distribution function [e.g., $f_0 = \frac{1}{\sqrt{2\pi}} e^{-\frac{v^2}{2}}$] is generated by N_p random numbers.

Key steps

Two key steps for PIC are: 1. Field $E(x_j)$ on grids to $E(x_i)$ on particle position; 2. Particle density $n(x_i)$ to grids $n(x_j)$. Suppose that the i-th electron lies between the j-th and $(j+1)$-th grid-points, i.e., $x_j < x_i \leq x_{j+1}$. Usually, the below interpolation method is used

$$n_j = n_j + \frac{x_{j+1} - x_i}{x_{j+1} - x_j} \frac{1}{\Delta x}, \tag{8a}$$

$$n_{j+1} = n_{j+1} + \frac{x_i - x_j}{x_{j+1} - x_j} \frac{1}{\Delta x}. \tag{8b}$$

The above procedure is repeated from the first particle to the last particle. Similar procedure are used to mapping $E(x_j)$ to $E(x_i)$.

PIC result

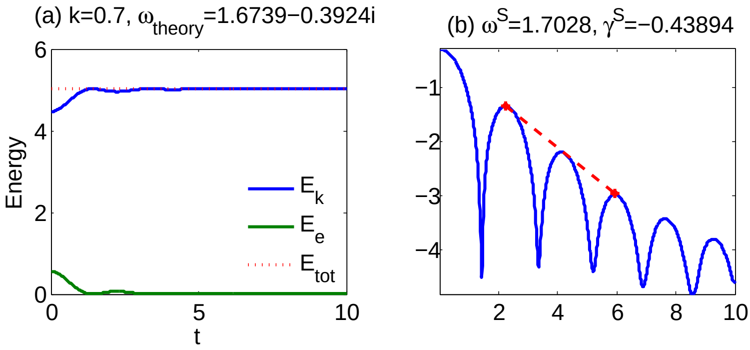

Figure 3: PIC simulation of Landau damping (pices1d.m code).

The energy conservation is very well. Real frequency and damping rate agree roughly with theory. A main drawback of PIC is the noise. Usually, very large N_p is required.

Vlasov continuity simulation: equations

Euler approach.
Vlasov equation

$$\partial_t f(x, v, t) = -v\partial_x f - \partial_x \phi \partial_v f, \tag{9}$$

Discrete

$$\frac{f_{i,j}^{n+1} - f_{i,j}^{n}}{\Delta t} = -v_j \frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2\Delta x} - \frac{\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n}}{2\Delta x} \frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2\Delta v}, \tag{10}$$

gives

$$f_{i,j}^{n+1} = f_{i,j}^{n} - v_j \frac{f_{i+1,j}^{n} - f_{i-1,j}^{n}}{2} \frac{\Delta t}{\Delta x} - \frac{\phi_{i+1,j}^{n} - \phi_{i-1,j}^{n}}{2\Delta x} \frac{f_{i,j+1}^{n} - f_{i,j-1}^{n}}{2} \frac{\Delta t}{\Delta v}. \tag{11}$$

Poisson equation

$$\partial_x^2 \phi = \int f dv - 1. \tag{12}$$

Discrete

$$\frac{\phi_{i+1} - 2\phi_i + \phi_{i-1}}{\Delta x^2} = \sum_j f_{i,j} \Delta v - 1 \equiv \rho_i, \qquad (13)$$

i.e.,

$$\begin{bmatrix} -2 & 1 & 0 & \cdot & \cdot & 0 & 1 \\ 1 & -2 & 1 & \cdot & \cdot & \cdot & 0 \\ 0 & 1 & -2 & 1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & 0 & \cdot & \cdot & \cdot & 1 & 2 \end{bmatrix} \begin{bmatrix} \phi_1 \\ \phi_2 \\ \cdot \\ \cdot \\ \phi_N \end{bmatrix} = \begin{bmatrix} \rho_1 \\ \rho_2 \\ \cdot \\ \cdot \\ \rho_N \end{bmatrix} \Delta x^2, \qquad (14)$$

where we have used the periodic boundary condition $\phi(0) = \phi(L)$, i.e., $\phi_1 = \phi_{N+1}$ and $\phi_0 = \phi_N$.

Simualtion results

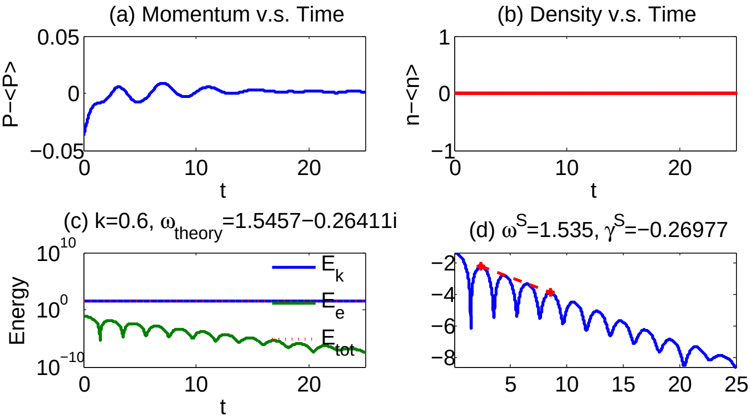

Figure 4: Vlasov continuity simulation, history plotting (code fkvl1d.m).

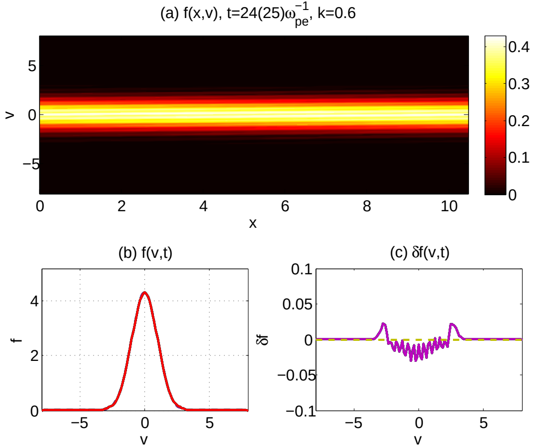

Figure 5: Vlasov continuity simulation, distribution function (code fkvl1d.m).

Nonlinear simulations

The PIC and Vlasov codes provided in the above sections can be easily modified to study the linear and nonlinear physics of the beam-plasma or two-stream instabilities.

A PIC simulation of two-stream instability is shown in Fig. 6 and Fig. 7. The linear growth and nonlinear saturation are very clear.

Exercise 2: Solving the kinetic or fluid dispersion relations for beam-plasma or two-stream plasma and comparing the results with linear and nonlinear simulations using the above models.

Simulation result

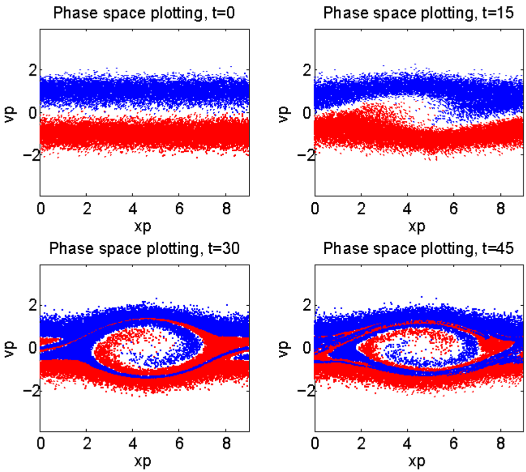

Figure 6: PIC simulation of the two-stream instability, phase space plotting.

Simulation result

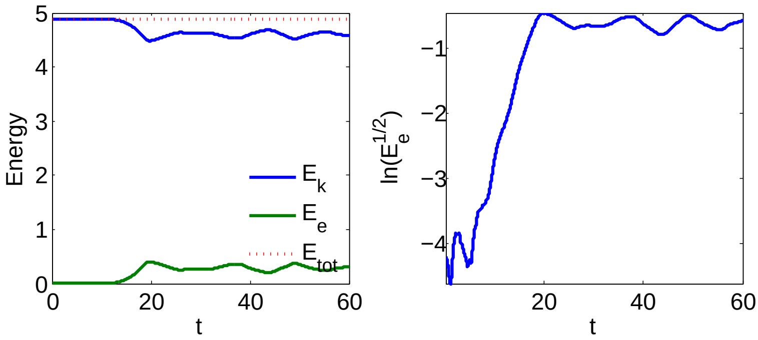

Figure 7: PIC simulation of the two-stream instability, history plotting.

C. Z. Cheng and G. Knorr, The integration of the vlasov equation in configuration space, Journal of Computational Physics, 22, 330 - 351, 1976.

H. S. Xie, Generalized Plasma Dispersion Function: One-Solve-All Treatment, Visualizations, and Application to Landau Damping, Phys. Plasmas, 20, 092125, 2013. Also, http://arxiv.org/abs/1305.6476. With codes.