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Guiding Center Particle Orbit in Tokamak

1. Equations of motion (White book, p70)

$$
\begin{cases}
\dot{\zeta} = \dfrac{\rho_{\|}B^2}{D}\left(q+\rho_{\|}I'\right)-\left(\mu+\rho_{\|}^2 B\right)\dfrac{I}{D}\dfrac{\partial B}{\partial \psi_p}-\dfrac{I}{D}\dfrac{\partial \Phi}{\partial \psi_p}, \\
\dot{\theta} = \dfrac{\rho_{\|}B^2}{D}\left(1-\rho_{\|}g'\right)+\left(\mu+\rho_{\|}^2 B\right)\dfrac{g}{D}\dfrac{\partial B}{\partial \psi_p}+\dfrac{g}{D}\dfrac{\partial \Phi}{\partial \psi_p}, \\
\dot{\psi}_p = -\dfrac{g}{D}\left(\mu+\rho_{\|}^2 B\right)\dfrac{\partial B}{\partial \theta}+\dfrac{I}{D}\left(\mu+\rho_{\|}^2 B\right)\dfrac{\partial B}{\partial \zeta}+\dfrac{I}{D}\dfrac{\partial \Phi}{\partial \zeta}-\dfrac{g}{D}\dfrac{\partial \Phi}{\partial \theta}, \\
\dot{\rho}_{\|} = -\dfrac{\left(1-\rho_{\|}g'\right)\left(\mu+\rho_{\|}^2 B\right)}{D}\dfrac{\partial B}{\partial \theta}-\dfrac{\left(1-\rho_{\|}g'\right)}{D}\dfrac{\partial \Phi}{\partial \theta} \\
\qquad -\dfrac{\left(q+\rho_{\|}I'\right)}{D}\dfrac{\partial \Phi}{\partial \zeta}-\dfrac{\left(q+\rho_{\|}I'\right)\left(\mu+\rho_{\|}^2 B\right)}{D}\dfrac{\partial B}{\partial \zeta}.
\end{cases}
$$

With $\rho_{\|}=v_{\|}/B, D=gq+I+\rho_{\|}\left(gI'_{\psi}-Ig'_{\psi}\right)$. And, $\mu=v_{\perp}^2/2B, E=\rho_{\|}^2B^2/2+\mu B+\Phi$.

$2\pi I$: toroidal current inside ψ. (p36)

$2\pi g$: poloidal current outside ψ.

$2\pi\psi$: toroidal flux. (p47)

$2\pi\psi_p$: poloidal flux. (p11)

2. Equilibrium (Reduce from Shafranov, White book, p47&48)

$d\psi/d\psi_p = q(\psi_p)$,

$\psi = r^2/2[1+O(\varepsilon^2)]$.

$R = 1 + r\cos\theta,\ r=\sqrt{2\psi},\ B=\sqrt{B_t^2+B_p^2},\ R_0=1$

$B_\phi \approx \dfrac{1}{R}$ (p52, eq2.90)

$B_\theta \approx \dfrac{r}{qR}$ (p51, eq2.87)

Or, $B_\theta \approx \dfrac{rB_\phi}{R_0 q} = \dfrac{rB_{\phi 0}}{Rq}$, dependent on θ.

$g = 1 + O(\varepsilon^2),\ g' = 0$

3. Constant q

$$I = \frac{r^2}{q},\ I' = \frac{2}{q} \quad \text{(p52)}$$

$$\psi_p = \psi / q$$

$$B' = \frac{B_t B_t' + B_p B_p'}{B}$$

$$\frac{\partial B}{\partial \psi_p} = \frac{B_t B_t' + B_p B_p'}{B} = \frac{\frac{g}{R}\frac{-g}{R^2}\frac{q\cos\theta}{r} + \frac{r}{qR}\frac{1}{q}\frac{qR/r - q\cos\theta r/r}{R^2}}{B}$$

$$= \frac{rR - r^2\cos\theta - g^2q^2\cos\theta}{BR^3rq} = \frac{r - g^2q^2\cos\theta}{BR^3rq},$$

$$\frac{\partial B}{\partial \theta} = \frac{\frac{g}{R}\frac{g}{R^2}r\sin\theta + \frac{r}{qR}\frac{r}{q}\frac{r\sin\theta}{R^2}}{B} = \frac{rB\sin\theta}{R}.$$

Test in **OrbitGC_qfix.m**.

4. q(psip)

$q(\psi_p) = q_1 + q_2\hat{\psi}_p + q_3\hat{\psi}_p^2$, [with $\hat{\psi}_p = \psi_p / \psi_w$ ψ_w at wall ($r = a$) for normalization].

$$\psi = \int q d\psi_p = \psi_p\left(q_1 + \frac{q_2}{2}\hat{\psi}_p + \frac{q_3}{3}\hat{\psi}_p^2\right)$$

$$r = \sqrt{2\psi} = \sqrt{2\psi_p\left(q_1 + \frac{q_2}{2}\hat{\psi}_p + \frac{q_3}{3}\hat{\psi}_p^2\right)} \quad \text{and} \quad a = \sqrt{2\psi_w\left(q_1 + \frac{q_2}{2} + \frac{q_3}{3}\right)}$$

$$I = \frac{r^2}{q(\psi_p)} = \frac{2\psi}{q(\psi_p)}, \text{ (p52)}$$

$$I'_\psi = \frac{2q - 2\psi q'_\psi}{q^2} = 2\frac{q - \psi q'_{\psi_p}/q}{q^2} = \frac{2}{q^3}\left[q^2 - \hat{\psi}_p(q_2 + 2q_3\hat{\psi}_p)(q_1 + \frac{q_2}{2}\hat{\psi}_p + \frac{q_3}{3}\hat{\psi}_p^2)\right]$$

$$r'_{\psi_p} = \frac{\left[\psi_p\left(q_1 + \frac{q_2}{2}\hat{\psi}_p + \frac{q_3}{3}\hat{\psi}_p^2\right)\right]'_{\psi_p}}{r} = \frac{q}{r}$$

$$B_t' = \frac{-gr'\cos\theta}{R^2} = \frac{-gq}{R^2 r}\cos\theta, \text{ [for } \psi_p \text{]}$$

$$B_p' = \frac{r'(qR) - r(qR)'}{(qR)^2} = \frac{r'q - rRq'}{(qR)^2} = \frac{q^2 / r - rR(q_2 + 2q_3\hat{\psi}_p)/\psi_w}{(qR)^2}, \text{ [for } \psi_p \text{]}$$

$$B' = \frac{B_t B_t' + B_p B_p'}{B}$$

$$\frac{\partial B}{\partial \psi_p} = \frac{B_t B_t' + B_p B_p'}{B} = \frac{\frac{g}{R}\frac{-g}{R^2}\frac{q\cos\theta}{r} + \frac{r}{qR}\frac{q^2 / r - rR(q_2 + 2q_3\hat{\psi}_p)/\psi_w}{(qR)^2}}{B}$$

$$= \frac{r\left[1 - \frac{r^2 R(q_2 + 2q_3\hat{\psi}_p)/\psi_w}{q^2}\right] - g^2 q^2 \cos\theta}{BR^3 rq},$$

$$\frac{\partial B}{\partial \theta} = \frac{\frac{g}{R}\frac{g}{R^2} r\sin\theta + \frac{r}{qR}\frac{r}{q}\frac{r\sin\theta}{R^2}}{B} = \frac{rB\sin\theta}{R}.$$

Test in **OrbitGC_qpsip.m**.

Appendix

Ref: C. Wrench, 2012, toroidal_coordinate_systems.pdf, sec8.3.1.

$$B_T = \frac{R_0 B_{T,0}}{R} = \frac{B_{T,0}}{1 + \varepsilon\cos\theta}, \ (\varepsilon = r / R_0)$$

$$B_P = \frac{1}{R}\frac{d\psi}{dr} = \frac{B_{P,0}}{1 + \varepsilon\cos\theta}.$$

$$q(r) = \frac{rB_{T,0}}{2\pi}\frac{dr}{d\psi}\int_0^{2\pi}\frac{d\theta}{1 + \varepsilon\cos\theta} = \frac{rB_{T,0}}{\sqrt{1 - \varepsilon^2}}\frac{dr}{d\psi}.$$

Define $q(r) = \frac{\bar{q}(r)}{\sqrt{1 - \varepsilon^2}}$

$$\frac{d\psi}{dr} = \frac{rB_{T,0}}{\bar{q}(r)}$$

$$\mathbf{B} = \frac{R_0 B_{T,0}}{R}\left[\mathbf{e}_\varphi + \frac{r}{\bar{q}R_0}\mathbf{e}_\theta\right],$$

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Lorentz Orbit in Tokamak

1. B field and coordinate

$$r=\sqrt{(\sqrt{x^2+y^2}-R_0)^2+z^2},$$

$$R=\sqrt{x^2+y^2},$$

q is general, e.g., $q=q_1+q_2r+q_3r^2$

$$B_t=\frac{B_0R_0}{R},$$

$$B_p=\frac{B_tr}{qR_0},$$

$$\begin{cases} B_x=B_t\dfrac{-y}{R}-B_p\dfrac{z}{r}\dfrac{x}{R}, \\ B_y=B_t\dfrac{x}{R}-B_p\dfrac{z}{r}\dfrac{y}{R}, \\ B_z=-B_p\dfrac{z}{r}\dfrac{R-R_0}{r}. \end{cases}$$

$$B=\sqrt{B_x^2+B_y^2+B_z^2}\ .$$

2. Equation of motion

$$\begin{cases} \dot{x}=v_x, \\ \dot{y}=v_y, \\ \dot{z}=v_z, \\ \dot{v}_x=\dfrac{q}{m}(E_x+v_yB_z-v_zB_y), \\ \dot{v}_y=\dfrac{q}{m}(E_y+v_zB_x-v_xB_z), \\ \dot{v}_z=\dfrac{q}{m}(E_z+v_xB_y-v_yB_x). \end{cases}$$

$v=\sqrt{K/m}$, K is kinetic energy, Λ is pitch angle.

$$\begin{cases} v_{\|}=v\cos\Lambda, \\ v_{\perp|}=v\sin\Lambda. \end{cases}$$

$$\begin{cases} v_x = (v_\parallel B_x + v_\perp B_x B_z / \sqrt{B_x^2 + B_y^2}) / B, \\ v_y = (v_\parallel B_y + v_\perp B_y B_z / \sqrt{B_x^2 + B_y^2}) / B, \\ v_z = (v_\parallel B_z - v_\perp \sqrt{B_x^2 + B_y^2}) / B. \end{cases}$$

$\mathbf{v} \cdot \mathbf{B} = v_\parallel B$ and $v = \sqrt{v_\parallel^2 + v_\perp^2} = \sqrt{v_x^2 + v_y^2 + v_z^2}$. The direction of $v_\perp$ is arbitrary. The above (v_x, v_y, v_z) is just one of them, which is used to set initial velocity.

Test in **Orbit_tokamak.m** and **Orbit.m**.