A Full-Matrix Approach to Solve Kinetic Plasma Dispersion Relation

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Ackn. (inspiration/discussion): GeFi&GTC projects, L. Chen, X. Y. Wang, Y. Lin, A. Bret, Ling Chen, X. R. Fu, ...

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1. Introduction

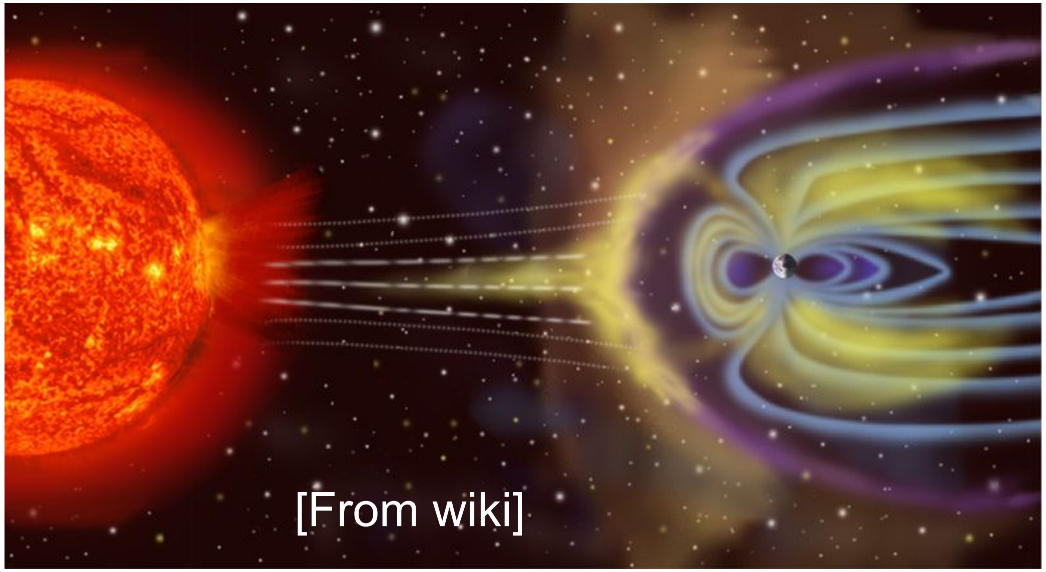
In: parameters (n, T, ...)
Out: all waves & instabilities in the system

1-1. Why linear dispersion relations

✓Richness waves & instabilities in astrophysical, space, laser, and laboratory plasmas.

✓Linear physics is a starting point.

✓The frontier is nonuniform & nonlinear, but still many difficult/important /interesting issues in linear problems!



Features:
- multi-species
- multi-scale
- kinetic
- beam
- anisotropic
- magnetized
- even relativistic
- nonuniform

1-2. Why this work (new solver)?

Hope: new tool & new physics

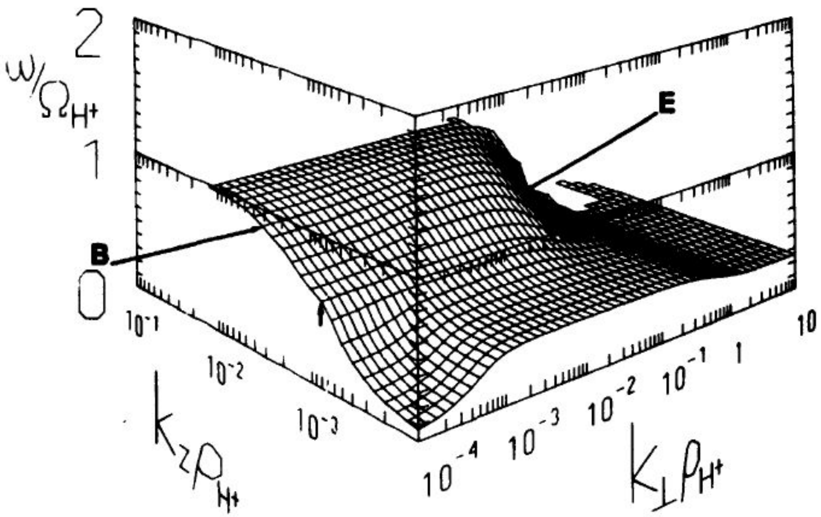
Conventional root finding (e.g., Newton's iteration):

1. Depends on initial guess, cannot show a completed picture, i.e., may missing important solutions.

2. Costly.

3. Singular -> unconvergent.

$$1-\sum_s \frac{\omega_{ps}^2}{\omega^2-\omega_{cs}^2}$$



high order cyclon-frequency (e.g., omega >~ 10*Omega_c) not well.

Fig: WHAMP result

1-3. Previous multi-fluid solver (PDRF*)

$$\lambda \mathbf{AX} = \mathbf{MX}$$

$$\partial_t n_j = -\nabla \cdot (n_j \mathbf{v}_j), \quad (1a)$$

$$\partial_t \mathbf{u}_j = -\mathbf{v}_j \cdot \nabla \mathbf{u}_j + \frac{q_j}{m_j}(\mathbf{E} + \mathbf{v}_j \times \mathbf{B}) - \frac{\nabla \mathbf{P}_j}{\rho_j} - \sum_i (\mathbf{u}_i - \mathbf{u}_j)\nu_{ij}, \quad (1b)$$

$$\partial_t \mathbf{E} = c^2 \nabla \times \mathbf{B} - \mathbf{J}/\epsilon_0, \quad (1c)$$

$$\partial_t \mathbf{B} = -\nabla \times \mathbf{E}, \quad (1d)$$

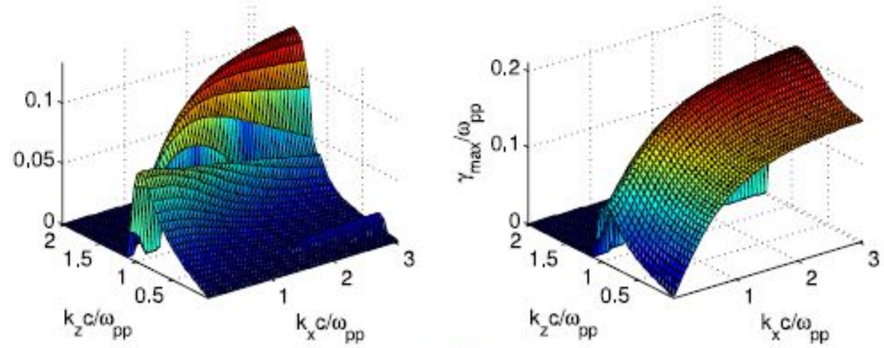
where $\mathbf{u}_j = \gamma_j \mathbf{v}_j$, and

$$\mathbf{J} = \sum_j q_j n_j \mathbf{v}_j, \quad (2a)$$

$$d_t (P_{\parallel j} \rho_j^{-\gamma_{\parallel j}}) = 0, \quad (2b)$$

$$d_t (P_{\perp j} \rho_j^{-\gamma_{\perp j}}) = 0, \quad (2c)$$

where $\rho_j \equiv m_j n_j$, $c^2 = 1/\mu_0 \epsilon_0$, $\gamma_j = (1 - v_j^2/c^2)^{-1/2}$, and $\gamma_{\parallel j}$ and $\gamma_{\perp j}$ are the parallel and perpendicular adiabatic coefficients, respectively. Furthermore, $P_{\parallel,\perp} = nT_{\parallel,\perp}$, $\mathbf{P} = P_\parallel \hat{\mathbf{b}}\hat{\mathbf{b}} + P_\perp (\mathbf{I} - \hat{\mathbf{b}}\hat{\mathbf{b}})$



(a) $B_0 \neq 0$. (b) $B_0 = 0$.

Fig. 3. The maximum growth rate γ_{max} vs. (k_x, k_z) for the relativistic electron beam mode with and without background B_0.

Fluid D.R. have been solved generally using full-matrix method previously.

$$\begin{bmatrix}
\{-i\mathbf{k}\cdot\mathbf{v}_{j0} & -ik_x n_{j0} - \epsilon_{njx} n_{j0} & -\epsilon_{njy} n_{j0} & -ik_z n_{j0} & 0 & 0 & 0 & 0 & 0 & 0 \\
\frac{-ik_x c_{\perp j}^2}{\rho_{j0}} & b_{jxx} & b_{jxy} + \omega_{cj} & b_{jxz} & \frac{q_j}{m_j} & 0 & 0 & -\frac{ik_z \Delta_j}{m_j n_{j0}} & -\frac{q_j v_{j0z}}{m_j} & \frac{q_j v_{j0y}}{m_j} \\
0 & b_{jyx} - \omega_{cj} & b_{jyy} & b_{jyz} & 0 & \frac{q_j}{m_j} & 0 & \frac{q_j v_{j0z}}{m_j} & -\frac{ik_z \Delta_j}{m_j n_{j0}} & -\frac{q_j v_{j0x}}{m_j} \\
\frac{-ik_z c_{\parallel j}^2}{\rho_{j0}} & b_{jzx} & b_{jzy} & b_{jzz}\} & 0 & 0 & \frac{q_j}{m_j} & -\frac{q_j v_{j0y}}{m_j} - \frac{ik_x \Delta_j}{m_j n_{j0}} & \frac{q_j v_{j0x}}{m_j} & 0 \\
-\frac{q_j v_{j0x}}{\epsilon_0} & -\frac{q_j n_{j0}}{\epsilon_0} & 0 & 0 & 0 & 0 & 0 & 0 & -ik_z c^2 & 0 \\
-\frac{q_j v_{j0y}}{\epsilon_0} & 0 & -\frac{q_j n_{j0}}{\epsilon_0} & 0 & 0 & 0 & 0 & ik_z c^2 & 0 & -ik_x c^2 \\
-\frac{q_j v_{j0z}}{\epsilon_0} & 0 & 0 & -\frac{q_j n_{j0}}{\epsilon_0} & 0 & 0 & 0 & 0 & ik_x c^2 & 0 \\
0 & 0 & 0 & 0 & 0 & ik_z & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -ik_z & 0 & ik_x & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -ik_x & 0 & 0 & 0 & 0
\end{bmatrix}$$

can obtain all accurate solutions in the system & not need derive D.R. under lots approx. any more.

*PDRF: A general dispersion relation solver for magnetized multi-fluid plasma, Computer Physics Communications, 2014, 185, 670 - 675.

2. Full-Matrix Approach

2-1. J-pole (Pade) Approx. of Z function*

$$F_M(v) = \frac{1}{\sqrt{\pi} v_t} e^{-\frac{v^2}{v_t^2}}$$

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_C \frac{e^{-z^2}}{z-\zeta} dz$$

Key step 1: J-pole for Z

$$Z(\zeta) = \sum_{j=1}^{J} \frac{b_j}{\zeta - c_j}$$

J=8

accurate in 3e-6
real axis

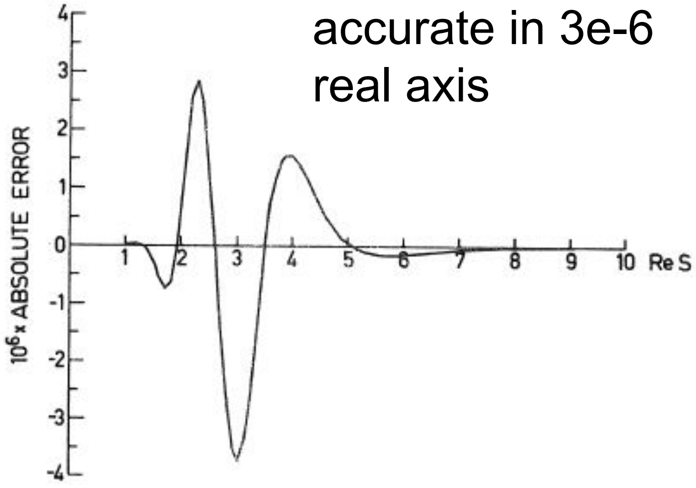


Figure 3 The absolute error in Im Z_A (s) versus s for Im s = 0.

* 1. Generalized Z function for almost **arbitrary distribution functions** is solved in [PoP, 2013, 20, 092125]
2. Scheme to calculate **arbitrary J** numerical coefficients also developed.

2-2. Electrostatic 1D

Key step 2: seek the equivalent linear transformation & matrix, i.e., Eq(4)

$$D = 1 + \sum_{s=1}^{S} \frac{1}{(k\lambda_{Ds})^2}[1 + \zeta_s Z(\zeta_s)] = 0 \qquad (2)$$

⇩

$$1 + \sum_s \sum_j \frac{b_{sj}}{(\omega - c_{sj})} = 0 \qquad b_{sj} = \frac{b_j c_j v_{ts}}{k\lambda_{Ds}^2} \text{ and } c_{sj} = k(v_{s0} + v_{ts} c_j). \qquad (3)$$

⇩

$$\omega v_{sj} = c_{sj} v_{sj} + b_{sj} E, \qquad (4a)$$

$$E = -\sum_{sj} v_{sj}, \qquad (4b)$$

$$f_{s0} = \left(\frac{m_s}{2\pi k_B T_s}\right)^{1/2} \exp\left[-\frac{(v - v_{s0})^2}{2k_B T_s}\right]$$

$$\lambda_{Ds}^2 = \frac{\epsilon_0 k_B T_s}{n_s q_s^2}, \; v_{ts} = \sqrt{\frac{2k_B T_s}{m_s}} \text{ and } \zeta_s = \frac{\omega - k v_{s0}}{k v_{ts}}$$

$$X = \{v_{sj}\}.$$

$SJ \times SJ$ dimensions eigen matrix M, i.e., $\omega X = MX$, with $SJ = S \times J$

- No singularity in (4).
- Standard matrix eigenvalue problem. Can be solved with no difficulty.
- Support multi-component easily and naturally.

1. Landau damping modes are accurately solved

Table 2: Comparing the Landau damping solutions using matrix method and original $Z(\zeta)$ function. Here, ω is normalized by $\omega_{pe} = \sqrt{n_e e^2/\epsilon_0 m_e}$.

$k\lambda_{De}$	$\omega_r^M(J=4)$	$\omega_i^M(J=4)$	$\omega_r^M(J=8)$	$\omega_i^M(J=8)$	$\omega_r^M(J=12)$	$\omega_i^M(J=12)$	ω_r^Z	ω_i^Z
0.1	0.9956	9.5E-3	1.0152	1.7E-5	1.0152	9.5E-8	1.0152	-4.8E-15
0.5	1.4235	-0.1699	1.4156	- 0.1534	1.4157	-0.1534	1.4157	-0.1534
1.0	2.0170	-0.8439	2.0459	- 0.8514	2.0458	-0.8513	2.0458	-0.8513
2.0	3.2948	- 2.6741	3.1893	- 2.8272	3.1891	-2.8272	3.1891	-2.8272

2. Electron bump-on-tail (s=e,b), both give (k=0.2): omega=0.9785+0.2000i

$T_b = T_e$, $v_b = 5v_{te}$ and $n_b = 0.1$

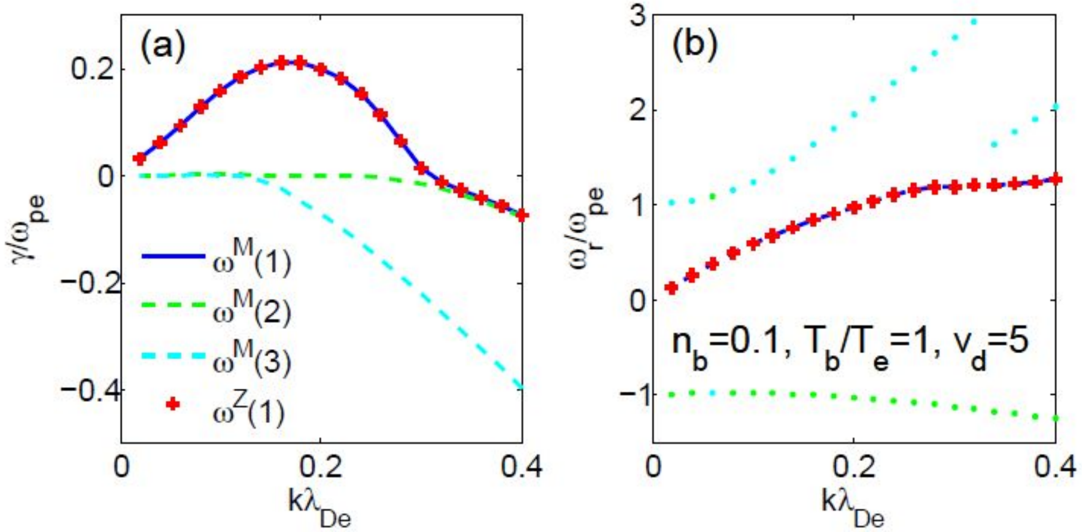


Figure 2: Comparing of the first three (ω^M) largest imaginary part solutions from matrix method ($J = 8$) and one solution (ω^Z) from $Z(\zeta)$ function for the bump-on-tail parameters.

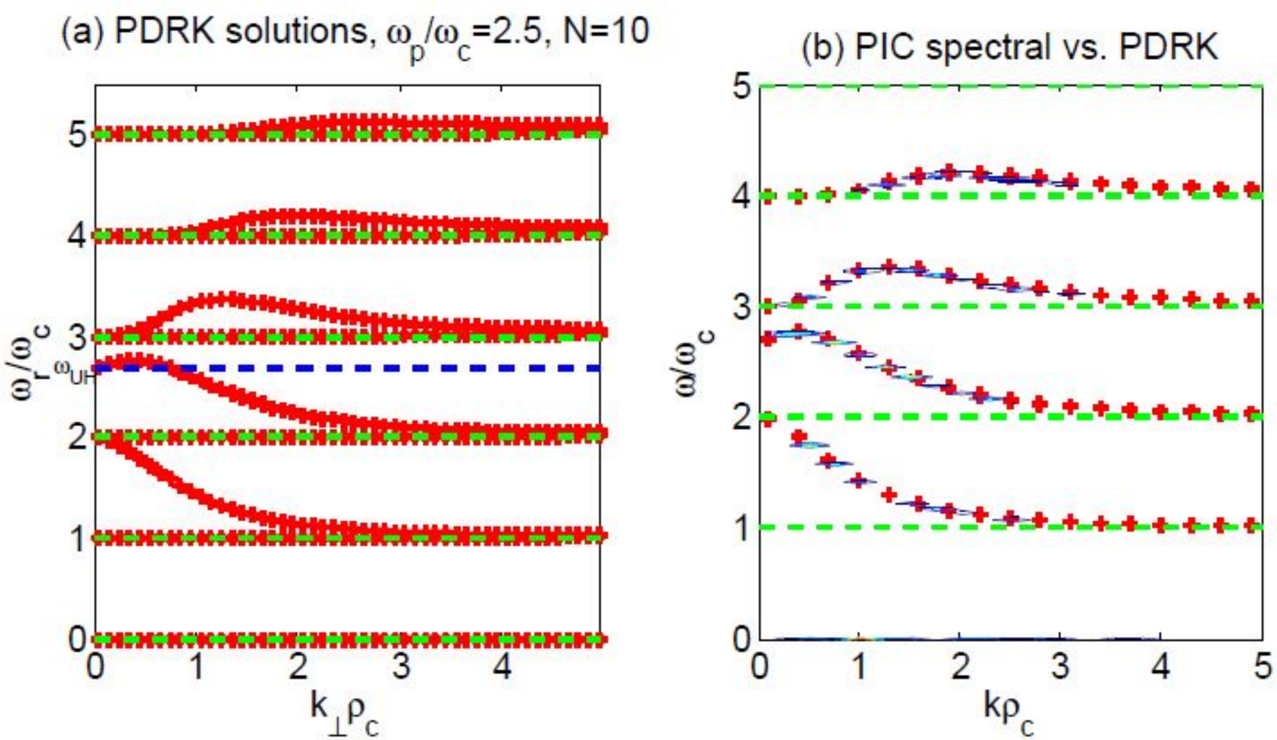
These give us confidence of the approach.

2-3. Harris dispersion relation (ES3D magnetized)

$$D = 1 + \sum_{s=1}^{S} \frac{1}{(k\lambda_{Ds})^2}[1 + \frac{\omega - k_z v_{s0} - n\Omega_s + \lambda_T n\Omega_s}{k_z v_{zts}} \sum_{n=-\infty}^{\infty} \Gamma_n(b_s)Z(\zeta_{sn})] = 0$$

1. Infinite order summation of Bessel functions

2. Transformation to equivalent linear system & matrix still straightforward. No difficulty.



N=10 is enough for lower order Bernstein mode.

Figure 3: The electron Bernstein modes calculated from Harris dispersion relation using matrix method. The upper hybrid frequency calculated at cold limit is $\omega_{UH} = \sqrt{\omega_c^2 + \omega_p^2} = 2.69$, which agrees with the matrix solution in the limit $k_\perp \rho_c \to 0$. The PDRK solutions also agree with the contour plot of the PIC spectral in (b).

2-4. EM3D drift bi-Maxwellian

1. The D.R.

$$\nabla\times\mathbf{B}=\mu_0\mathbf{J}+\epsilon_0\mu_0\frac{\partial\mathbf{E}}{\partial t}$$
$$\nabla\times\mathbf{E}=-\frac{\partial\mathbf{B}}{\partial t}$$
$$\nabla\cdot\mathbf{E}=\frac{\rho_q}{\epsilon_0}$$
$$\nabla\cdot\mathbf{B}=0$$

$$\mathrm{i}\mathbf{k}\times\tilde{\mathbf{B}}=\epsilon_0\mu_0(-\mathrm{i}\omega)\overleftrightarrow{\mathbf{K}}\cdot\tilde{\mathbf{E}}.$$

$$\tilde{\mathbf{J}}=\overleftrightarrow{\sigma}\cdot\tilde{\mathbf{E}},$$

$$\overleftrightarrow{\mathbf{K}}=\overleftrightarrow{\mathbf{1}}-\frac{\overleftrightarrow{\sigma}}{\mathrm{i}\omega\epsilon_0}$$

$$\begin{bmatrix} K_{xx}-\frac{c^2k^2}{\omega^2}\cos^2\theta & K_{xy} & k_{xz}+\frac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ K_{yx} & K_{yy}-\frac{c^2k^2}{\omega^2} & K_{yz} \\ K_{zx}+\frac{c^2k^2}{\omega^2}\sin\theta\cos\theta & K_{zy} & K_{zz}-\frac{c^2k^2}{\omega^2}\sin\theta \end{bmatrix}\begin{bmatrix}\tilde{E}_x\\ \tilde{E}_y\\ \tilde{E}_z\end{bmatrix}=0$$

$$\boldsymbol{K}=\boldsymbol{I}+ \sum_{\mathrm{i,e}}\frac{\Pi^2}{\omega^2}\left[\sum_n\left\{\zeta_0Z(\zeta_n)-\left(1-\frac{1}{\lambda_T}\right)[1+\zeta_nZ(\zeta_n)]\right\}\mathrm{e}^{-b}\boldsymbol{X}_n+2\eta_0^2\lambda_T\boldsymbol{L}\right], \tag{12.19}$$

$$\boldsymbol{X}_n=\begin{pmatrix} n^2I_n/b & \mathrm{i}n(I_n'-I_n) & -(2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}I_n \\ -\mathrm{i}n(I_n'-I_n) & (n^2/b+2b)I_n-2bI_n' & \mathrm{i}(2\lambda_T)^{1/2}\eta_n\alpha(I_n'-I_n) \\ -(2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}I_n & -\mathrm{i}(2\lambda_T)^{1/2}\eta_n\alpha(I_n'-I_n) & 2\lambda_T\eta_n^2I_n \end{pmatrix}, \tag{12.20}$$

Maxwell equation doesn't need change. For simplicity, we just need to seek an equivalent relation of J(E).

2. The transformation

E still electric field. The eigenvectors represent the polarizations as in fluid.

$$\begin{bmatrix} J_x \\ J_y \\ J_z \end{bmatrix} = \begin{pmatrix} a_{11} - \sum_{snj} \frac{b_{snj11}}{\lambda - c_{snj11}} & a_{12} - \sum_{snj} \frac{b_{snj12}}{\lambda - c_{snj12}} & a_{13} - \sum_{snj} \frac{b_{snj13}}{\lambda - c_{snj13}} \\ a_{21} - \sum_{snj} \frac{b_{snj21}}{\lambda - c_{snj21}} & a_{22} - \sum_{snj} \frac{b_{snj22}}{\lambda - c_{snj22}} & a_{23} - \sum_{snj} \frac{b_{snj23}}{\lambda - c_{snj23}} \\ a_{31} - \sum_{snj} \frac{b_{snj31}}{\lambda - c_{snj31}} & a_{32} - \sum_{snj} \frac{b_{snj32}}{\lambda - c_{snj32}} & a_{33} - \sum_{snj} \frac{b_{snj33}}{\lambda - c_{snj33}} + d_{33}\lambda \end{pmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}$$

$$\begin{cases} \omega v_{snjx} &= c_{snj} v_{snjx} + b_{snj11} E_x + b_{snj12} E_y + b_{snj13} E_z, \\ \omega j_x &= b_{11} E_x + b_{12} E_y + b_{13} E_z, \\ J_x &= j_x + \sum_{snj} v_{snjx}, \\ \omega v_{snjy} &= c_{snj} v_{snjy} + b_{snj21} E_x + b_{snj22} E_y + b_{snj23} E_z, \\ \omega j_y &= b_{21} E_x + b_{22} E_y + b_{23} E_z, \\ J_y &= j_y + \sum_{snj} v_{snjy}, \\ \omega v_{snjz} &= c_{snj} v_{snjz} + b_{snj31} E_x + b_{snj32} E_y + b_{snj33} E_z, \\ \omega j_z &= b_{31} E_x + b_{32} E_y + b_{33} E_z, \\ J_z &= j_z + \sum_{snj} v_{snjz}, \\ \omega E_x &= -c^2 k_z B_y - J_x/\epsilon_0, \\ \omega E_y &= c^2 k_z B_x - c^2 k_x B_z - J_y/\epsilon_0, \\ \omega E_z &= c^2 k_x B_y - J_z/\epsilon_0, \\ \omega B_x &= k_z E_y, \\ \omega B_y &= -k_z E_x + k_x E_z, \\ \omega B_z &= -k_z E_y, \end{cases}$$

1. Thus, simple and matrix be sparse!

2. If d_33 $\neq$0, transformation will complicated, whereas a_xyz $\neq$ 0 still straghtforward.

3. If matrix not sparse: memory O(NN^2), CPU time O(NN^3). Single PC support NN~6000 in minutes with ~1G memory. NN can be 10^5 for sparse matrix.

SNJ=3*[S*(2*N+1)*J+1];
NN=SNJ+6;

S=2, N=50, J=8 => NN~6000

3. PDRK: Benchmarks & Applications

3-1. Benchmarks

1. PDRK (dot) vs. PDRF (solid line), cold (Te = Ti = 0.01), parallel propagation

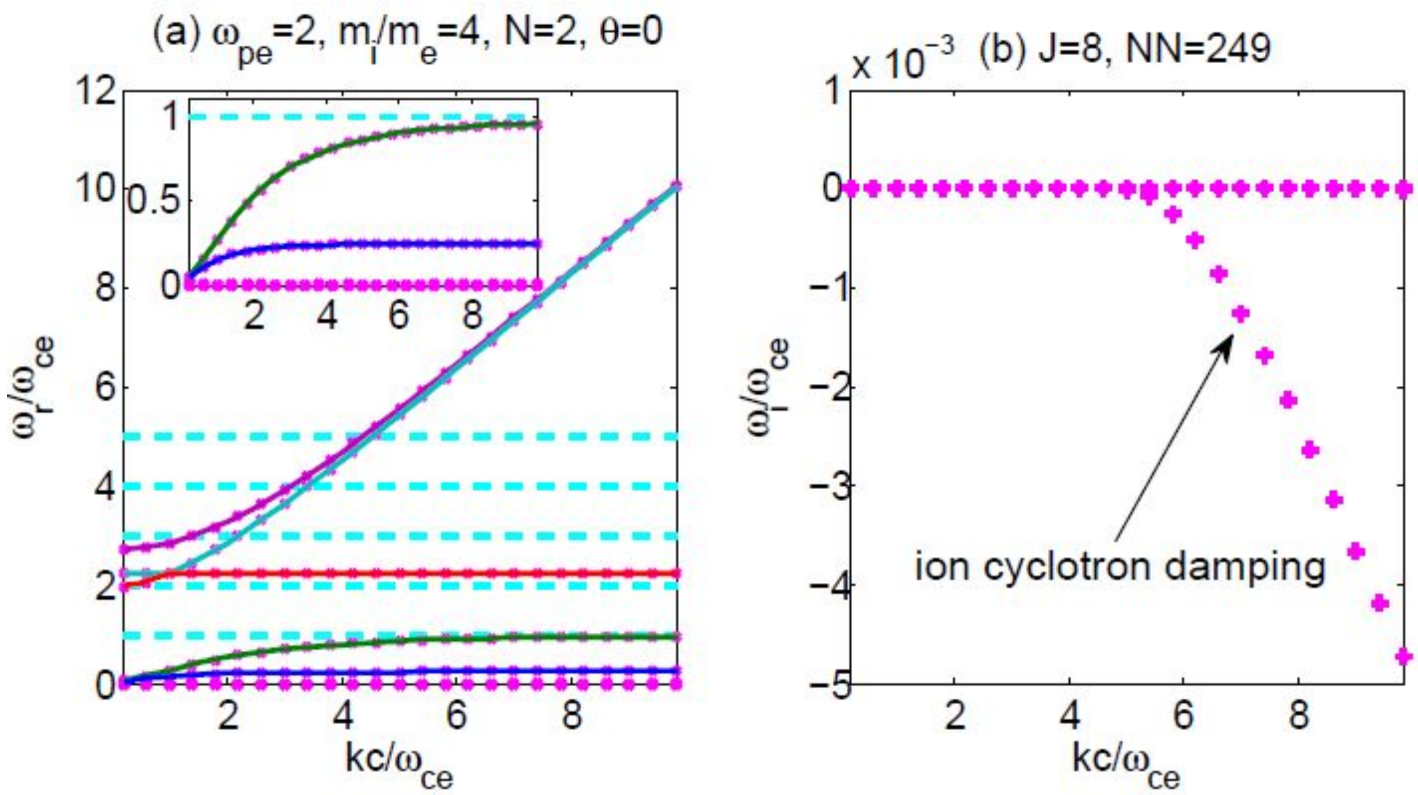


Figure 5: PDRK (dot) vs. PDRF (solid line), cold ($T_e = T_i = 0.01$), parallel propagation.

no cyclotron damping in fluid

2. PDRK (dot) vs. PDRF (solid line), warm (Te = Ti = 100), perpendicular propagation

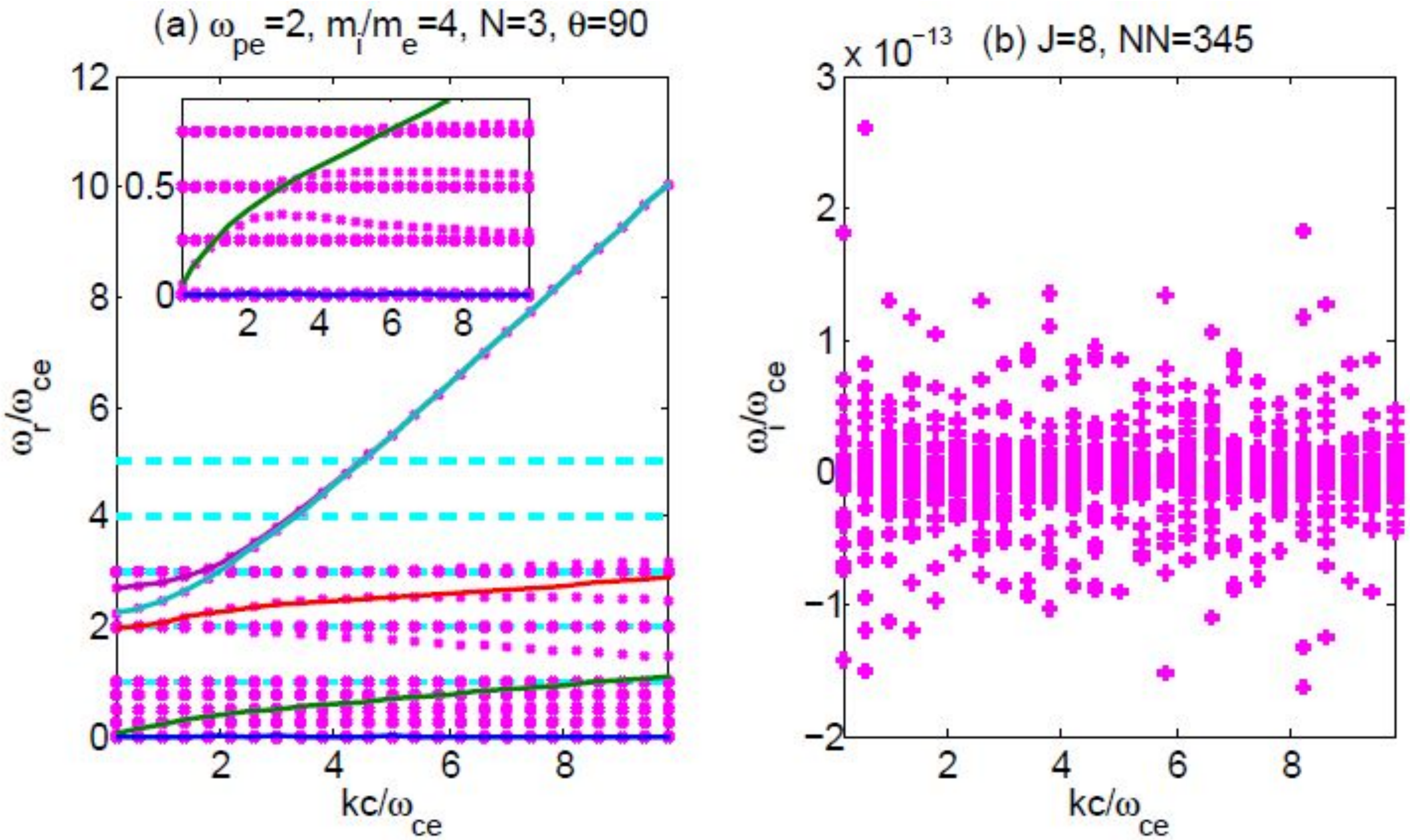


Figure 6: PDRK (dot) vs. PDRF (solid line), warm ($T_e = T_i = 100$), perpendicular propagation. The positive $\gamma \simeq 10^{-13}$ comes from numerical error of $J = 8$.

no Bernstein modes in fluid

3. Parallel propagation kinetic modes

$$D(k,\omega) = 1 - \frac{k^2c^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2}{\omega k v_{ts}} Z\left(\frac{\omega \pm \omega_{cs}}{k v_{ts}}\right) = 0$$

parallel propagation kinetic dispersion relation, is relatively simple to be solved, J_n=0

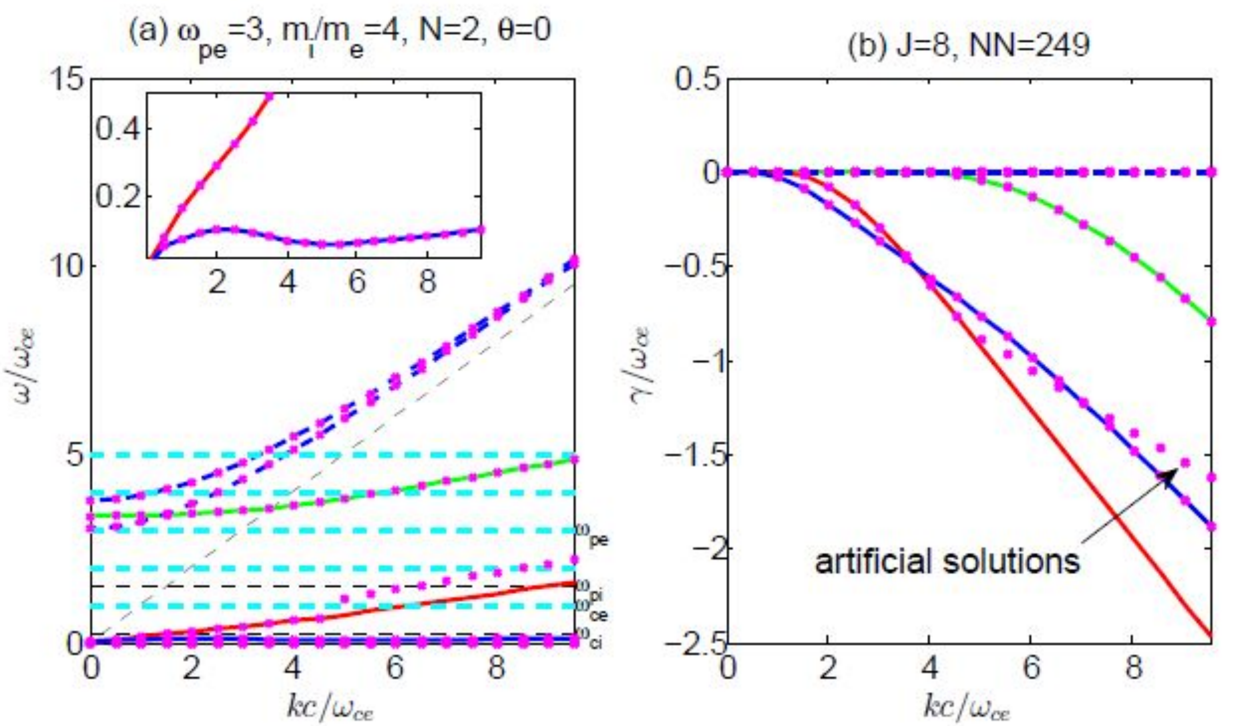


Figure 7: PDRK solutions (dot) vs. Z function solutions (solid and green dash lines), warm ($T_e = T_i = 400$), parallel propagation. Heavily damped (both real and artificial) solutions are not shown.

Agree well, but many (heavily damped) artificial solutions in PDRK

4. Landau damping of lower hybrid wave

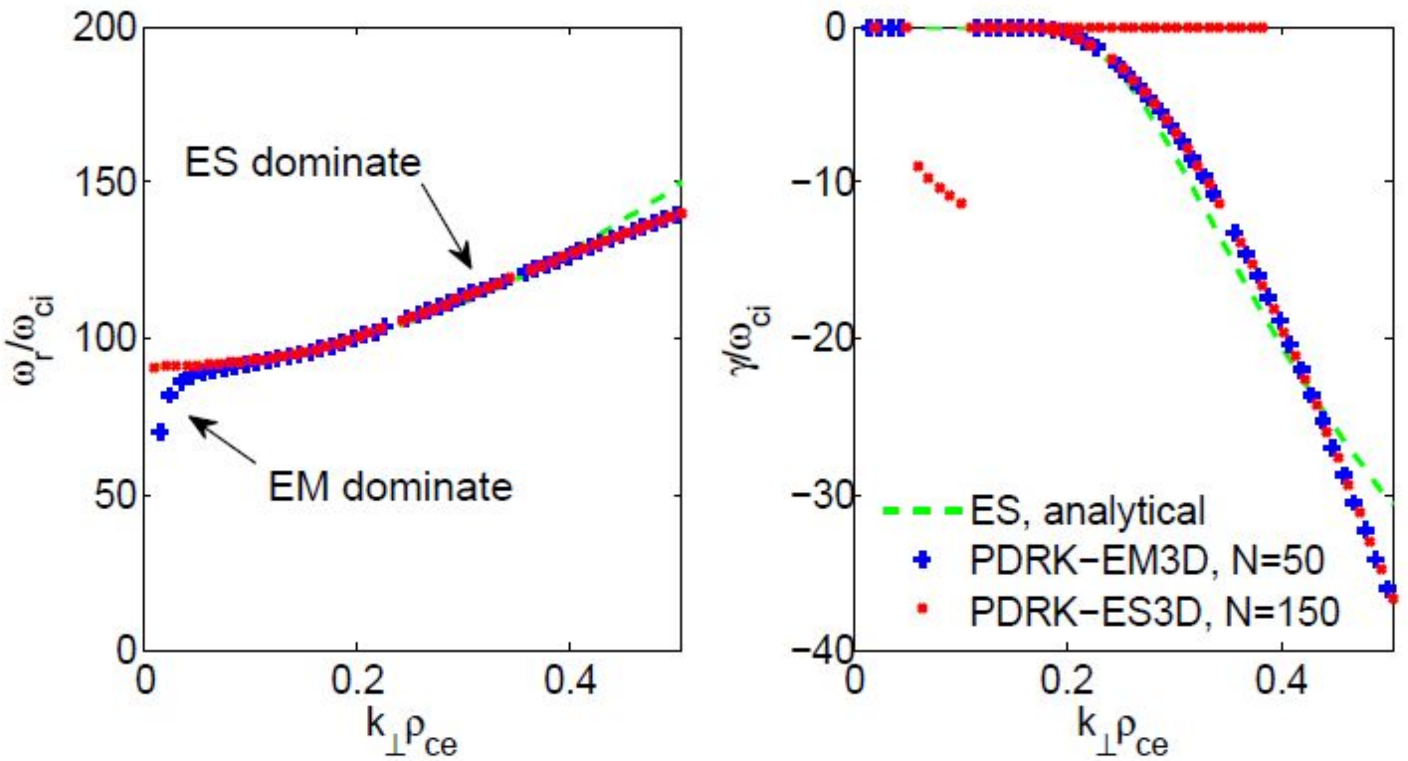


Figure 8: Landau damping of lower hybrid wave. Solutions from PDRK-ES3D (red, $N = 150$), PDRK-EM3D (blue, $N = 50$), and analytical solution (dash green line) in Ref.[20]. About 1 CPU hour is taken to compute the data in this figure.

exact EM-LHW is not easy to be solved in convetional solvers

5. Firehose and mirror modes

anisotropic

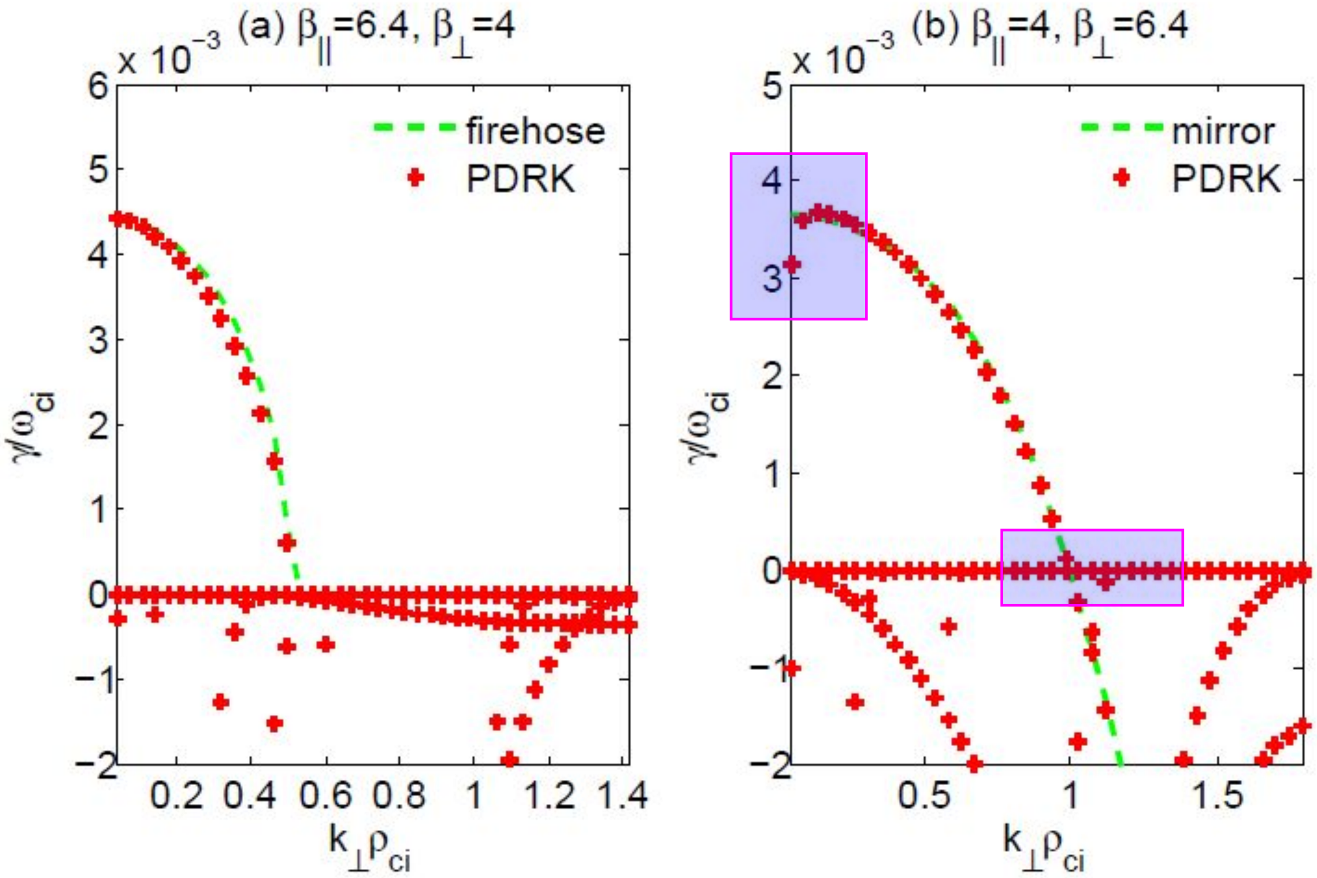


Figure 9: Growth rates for firehose and mirror modes v.s. $k_\perp\rho_{ci}$. The dash green lines are analytical solutions.

$$\omega^2 = \omega_A^2\left[\frac{b_i}{1-\Gamma_0(b_i)} + \frac{\beta_{i\perp}-\beta_{i\|}}{2}\right]$$

$$\zeta_i Z(\zeta_i) = \frac{\eta_i}{\beta_{i\perp}\Gamma_1(b_i)} - (1-\eta_i)$$

Firehose and mirror modes are typical unstable modes driven by pressure anisotropic $T_\| \neq T_\perp$.

6. Whistler beam mode

beam

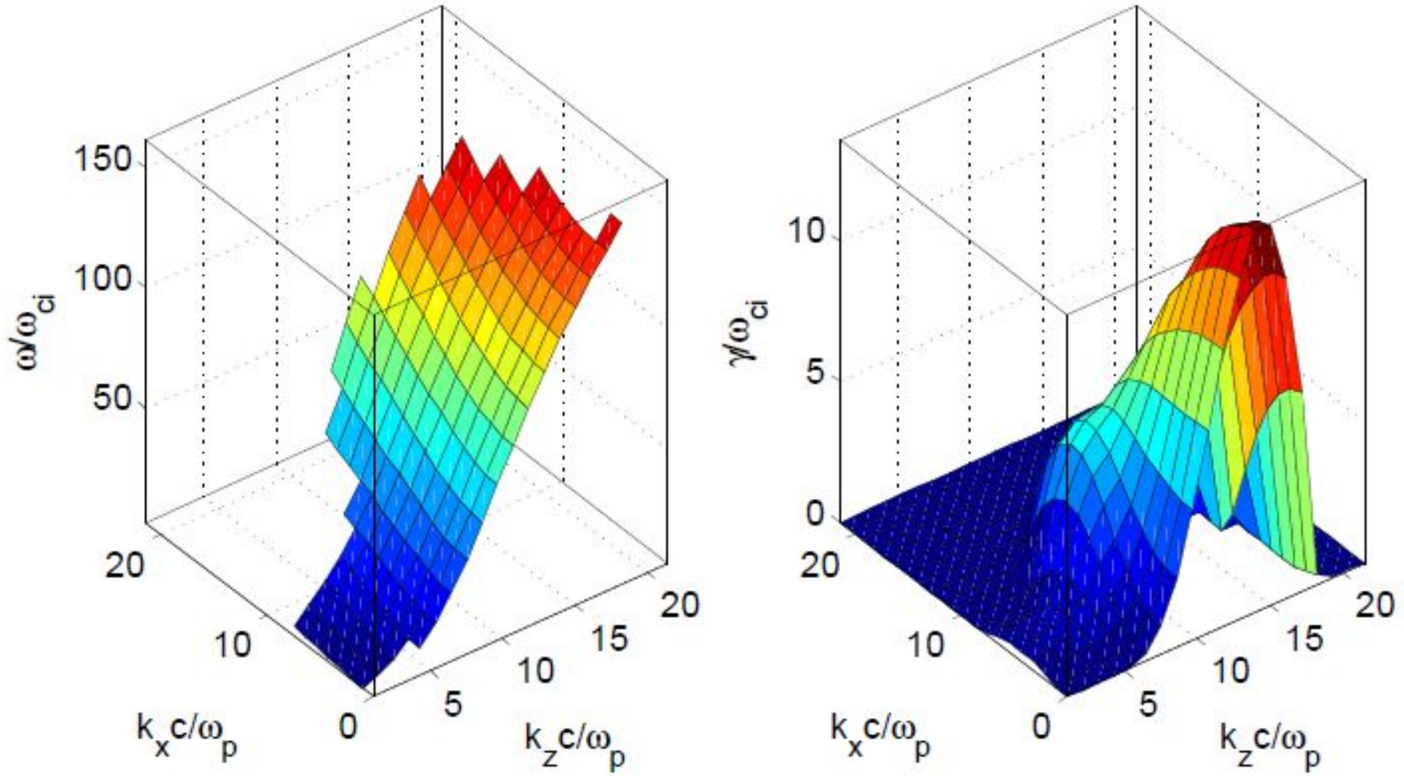
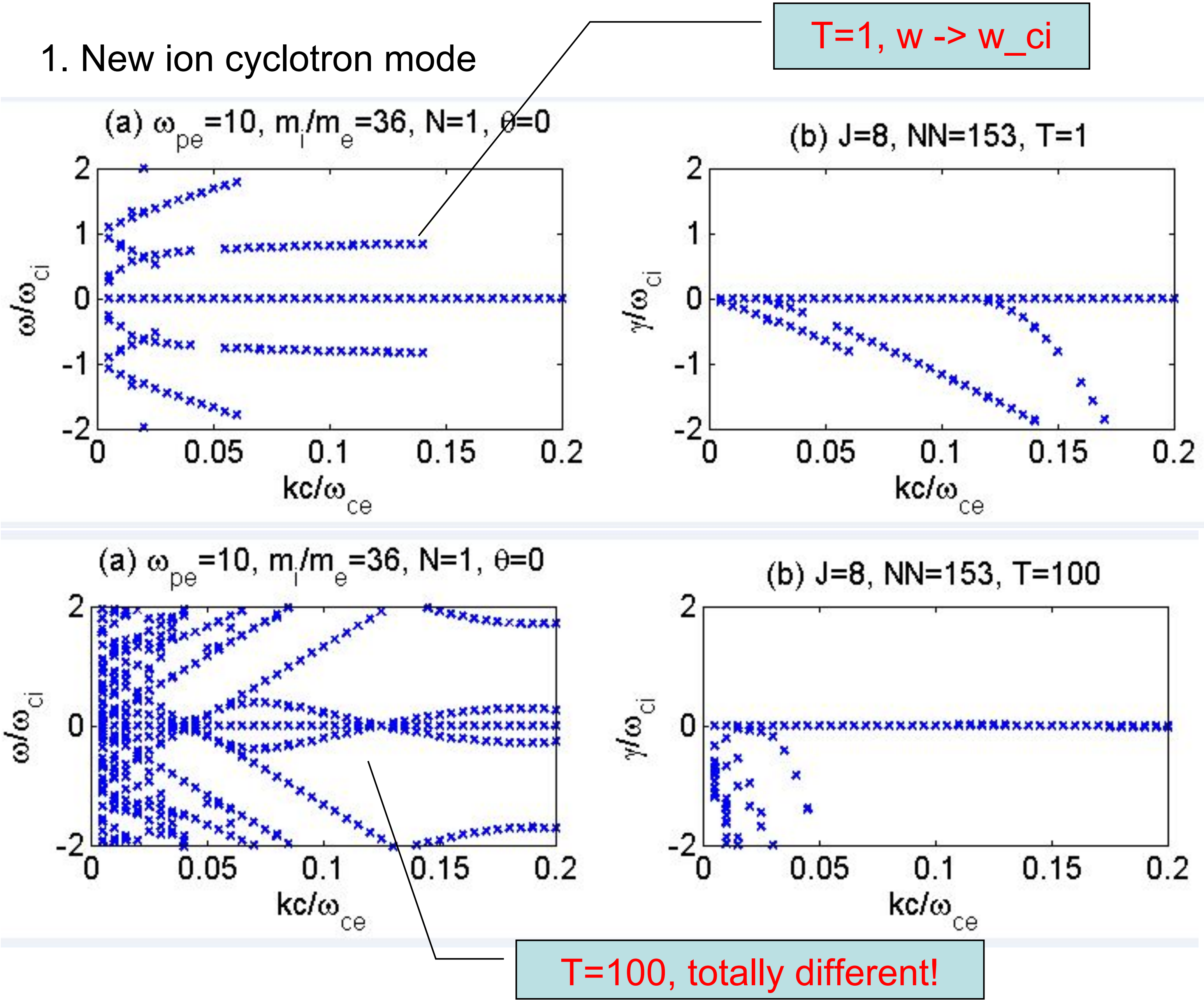


Figure 10: Electromagnetic whistler beam instability. The real frequency ω is only shown for unstable ($\gamma > 0$) solutions. The parallel propagation ($k = k_{\parallel}$) results are similar to Fig.8.8 of Ref.[5]. $N = 3$ is used for this calculation.

most unstable mode is parallel propagation mode

3-2. Applications

1. New ion cyclotron mode

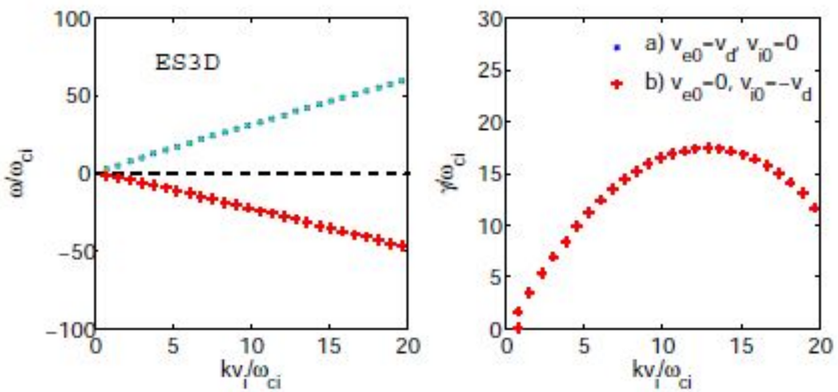


2. New **anomalous Doppler** shift

non-relativistic, i.e., vts, vd < 0.01c << c

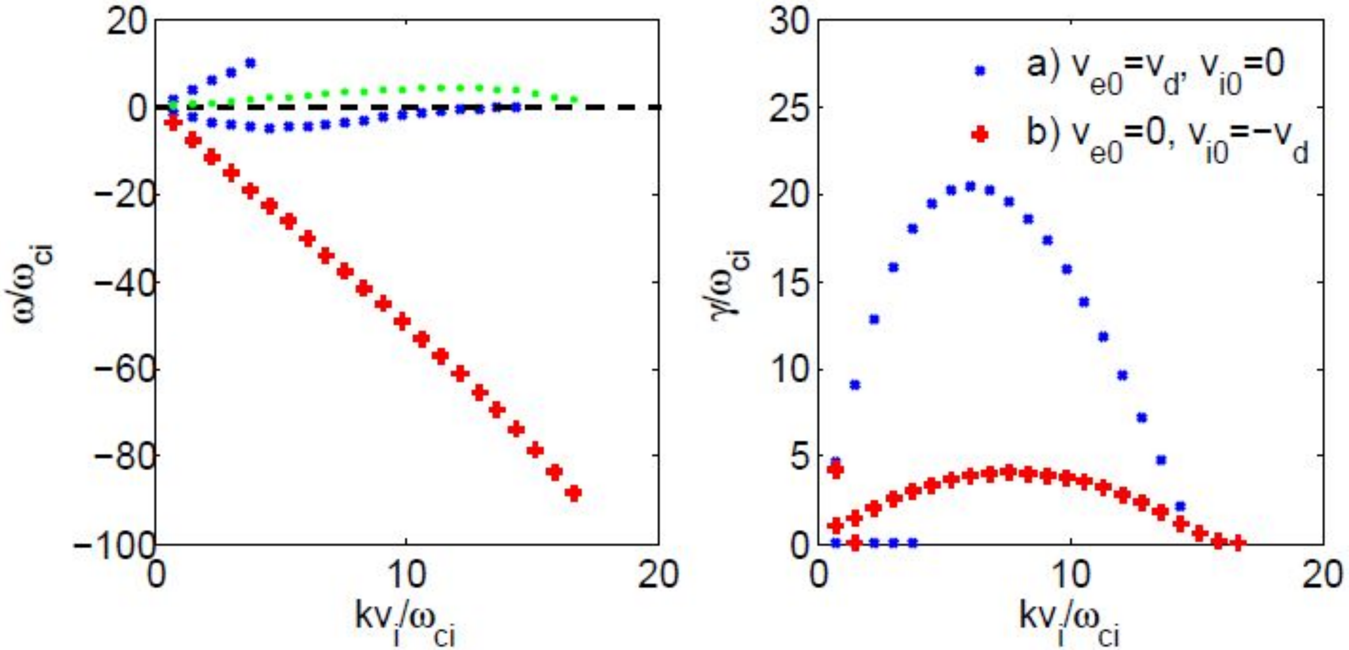
$$\mathbf{k}' = \mathbf{k}, \qquad \omega' = \omega - \mathbf{k}\cdot\mathbf{v}_0,$$

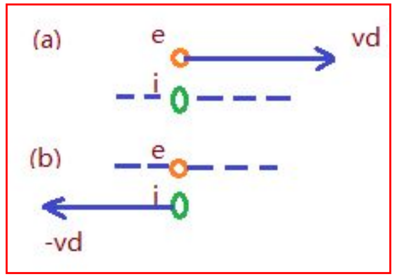
$$\mathbf{k}' = \gamma(\mathbf{k}_\parallel - \mathbf{v}_0\omega/c^2) + \mathbf{k}_\perp, \qquad \omega' = \gamma(\omega - \mathbf{k}\cdot\mathbf{v}_0),$$



ES3D is Galilean

asymmetry of electron and ion beams





EM not Galilean invariant!!

Figure 11: New modes found by PDRK-EM3D: the Doppler asymmetry of electron and ion beams in kinetic non-relativistic plasmas. Blue '×' for a); red '+' for b); green dot is $\omega = \omega^b - k_\parallel v_d$.

Similar results are found by **GeFi** (Wei KONG) and **pic-em1d3v** simulations and analytical result.

4. Summary and Comments

1. A general, fast, and effective approach is developed to numerically solve kinetic plasma dispersion relations, which has avioded singularity & time-costly difficulties, especially, all important solutions can be obtained (won't miss solutions).

2. Can identify validity of simplied models, e.g., Darwin, gyro-kinetic, fluid.

3. Wide applications & new modes could be found.

4. Other distributions, new J-pole straightforward.

5. Need remove extraneous solutions.

6. Need relativistic version.

7. Next step: nonuniform global eigenvalue (Tokamak)