Detailed Derivations of PDRK-EM3D Equations

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The PDRK-EM3D code is found disagree with WHAMP code for hot plasma $k_\perp \neq 0$ modes. This note redo the detailed derivations and to check where come the differences/bugs.

See also 2018-06-13 draft.

Refs:

[Xie2016] Huasheng Xie and Yong Xiao, PDRK: A General Kinetic Dispersion Relation Solver for Magnetized Plasma, Plasma Science and Technology, 18, 2, 97 (2016). http://hsxie.me/codes/pdrk/

[Miyamoto2004] Kenro Miyamoto, Plasma Physics and Controlled Nuclear Fusion, Springer, 2004.

[Gary1993] S. Peter Gary, Theory of Space Plasma Microinstabilities, Cambridge, 1993.

1 PDRK Equation

The below equations come from the PDRK original paper with fixed several typos (highlighted with red color) and are used in PDRK code.

1.1 Notations

The disagreement is likely from the Bessel function term relevant to $k_\perp\rho_{ts}$, and which is related to the definition of v_{ts}, e.g., $v_{ts}=\sqrt{\frac{2k_BT_s}{m_s}}$ v.s. $v_{ts}=\sqrt{\frac{k_BT_s}{m_s}}$.

1.1.1 ES3D

The notations for ES3D in the PDRK paper [Xie2016] are $\lambda_{Ds}^2=\frac{\epsilon_0 k_BT_{zs}}{n_{s0}q_s^2}$, $v_{ts}=\sqrt{\frac{2k_BT_s}{m_s}}$, $\lambda_T=T_z/T_\perp$, $\zeta_{sn}=\frac{\omega-k_zv_{s0}-n\Omega_s}{k_zv_{zts}}$, $\Gamma_n(b)=I_n(b)e^{-b}$, $b_s=k_\perp^2\rho_{cs}^2$, $\rho_{cs}=\frac{v_{\perp ts}}{\sqrt{2}\Omega_s}$ [to check], I_n is the modified Bessel function, and the equilibrium distribution function is assumed to be drift bi-Maxwellian distribution $f_{s0}=n_{s0}f_\perp(v_\perp)f_z(v_z)$, with $f_\perp=\frac{m_s}{2\pi k_BT_{s\perp}}\exp[-\frac{m_sv_\perp^2}{2k_BT_{s\perp}}]$ and $f_z=(\frac{m_s}{2\pi k_BT_{sz}})^{1/2}\exp[-\frac{m_s(v_\parallel-v_{s0})^2}{2k_BT_{sz}}]$.

1.1.2 EM3D

We have noticed that the actual definition of some notations for PDRK-EM3D and ES3D are not the same in the original paper.

The actual notations[1] for EM3D in the paper are $\lambda_{Ds}^2 = \frac{\epsilon_0 k_B T_{zs}}{n_{s0} q_s^2}$, $v_{ts} = \sqrt{\frac{k_B T_s}{m_s}}$, $\lambda_T = T_z/T_\perp$, $\zeta_{sn} = \frac{\omega - k_z v_{s0} - n\Omega_s}{\sqrt{2} k_z v_{zts}}$, $\Gamma_n(b) = I_n(b)e^{-b}$, $b_s = k_\perp^2 \rho_{cs}^2$, $\rho_{cs} = \sqrt{\frac{k_B T_{s\perp}}{m_s}} \frac{1}{\Omega_s} = \frac{v_{\perp ts}}{\Omega_s}$, I_n is the modified Bessel function, and the equilibrium distribution function is assumed to be drift bi-Maxwellian distribution $f_{s0} = n_{s0} f_\perp(v_\perp) f_z(v_z)$, with $f_\perp = \frac{m_s}{2\pi k_B T_{s\perp}} \exp[-\frac{m_s v_\perp^2}{2k_B T_{s\perp}}]$ and $f_z = (\frac{m_s}{2\pi k_B T_{sz}})^{1/2} \exp[-\frac{m_s (v_\parallel - v_{s0})^2}{2k_B T_{sz}}]$.

Missed definitions in original paper[2]: $\omega_{ps}^2 = \frac{n_{s0} q_s^2}{\epsilon_0 m_s}$, $\Omega_s = \frac{q_s B_0}{m_s}$.

1.2 EM3D Dispersion relation

The background magnetic field is assumed to be $\boldsymbol{B}_0 = (0, 0, B_0)$, and the wave vector $\boldsymbol{k} = (k_x, 0, k_z) = (k\sin\theta, 0, k\cos\theta)$, which gives $k_\perp = k_x$ and $k_\parallel = k_z$.

The dispersion relation is [Stix1992, Miyamoto2004]

$$\begin{vmatrix} K_{xx} - \frac{c^2k^2}{\omega^2}\cos^2\theta & K_{xy} & K_{xz} + \frac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ K_{yx} & K_{yy} - \frac{c^2k^2}{\omega^2} & K_{yz} \\ K_{zx} + \frac{c^2k^2}{\omega^2}\sin\theta\cos\theta & K_{zy} & K_{zz} - \frac{c^2k^2}{\omega^2}\sin^2\theta \end{vmatrix} = 0, \quad (1)$$

with $\boldsymbol{K} = \boldsymbol{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2}\Big[\sum_n \big\{\zeta_0 Z(\zeta_n) - (1 - \frac{1}{\lambda_T})[1 + \zeta_n Z(\zeta_n)]\big\}\boldsymbol{X}_n + 2\eta_0^2\lambda_T \boldsymbol{L}\Big]$, where in the original paper [Xie2016]

$$\boldsymbol{X}_n = \begin{pmatrix} n^2\Gamma_n/b & in\Gamma_n' & -(2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}\Gamma_n \\ -in\Gamma_n' & n^2\Gamma_n/b - 2b\Gamma_n' & i(2\lambda_T)^{1/2}\eta_n\alpha\Gamma_n' \\ -(2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}\Gamma_n & -i(2\lambda_T)^{1/2}\eta_n\alpha\Gamma_n' & 2\lambda_T\eta_n^2\Gamma_n \end{pmatrix}, \quad (2)$$

and the correction one should be

$$\boldsymbol{X}_n = \begin{pmatrix} n^2\Gamma_n/b & in\Gamma_n' & (2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}\Gamma_n \\ -in\Gamma_n' & n^2\Gamma_n/b - 2b\Gamma_n' & -i(2\lambda_T)^{1/2}\eta_n\alpha\Gamma_n' \\ (2\lambda_T)^{1/2}\eta_n\frac{n}{\alpha}\Gamma_n & i(2\lambda_T)^{1/2}\eta_n\alpha\Gamma_n' & 2\lambda_T\eta_n^2\Gamma_n \end{pmatrix}, \quad (3)$$

with $\eta_n = \frac{\omega - n\Omega}{\sqrt{2}k_z v_{T_z}}$, $\lambda_T = \frac{T_z}{T_\perp}$, $b = (\frac{k_x v_{T\perp}}{\Omega})^2$, $\alpha = \frac{k_x v_{T\perp}}{\Omega}$, $v_{T_z}^2 = \frac{k_B T_z}{m}$, $v_{T_\perp}^2 = \frac{k_B T_\perp}{m}$, and the matrix components of $\boldsymbol{L}$ are all zero, except for $L_{zz} = 1$.

Note: $\Gamma_n'(b) = (I_n' - I_n)e^{-b}$, $I_n'(b) = (I_{n+1} + I_{n-1})/2$, $I_{-n} = I_n$.

18-09-28 11:50 The original sign of $\boldsymbol{X}_{n13}$, $\boldsymbol{X}_{n23}$, $\boldsymbol{X}_{n31}$ and $\boldsymbol{X}_{n32}$ are opposite and have been corrected in this version. This is due to the misleading definition of $\Omega_s = -\frac{q_s B}{m_s}$ in Miyamoto2004, which affects the α term. The sign of $n\Omega$ in η_n and ζ_n has also been corrected, i.e., $\eta_n = \frac{\omega + n\Omega_s}{\sqrt{2}k_z v_{Tz}}$ and $\zeta_n = \frac{\omega - k_z v_{s0} + n\Omega_s}{\sqrt{2}k_z v_{zts}}$ in Miyamoto2004.

[1] These notations are not written explicitly in the original paper, which should be mainly from Miyamoto2004.

[2] In Miyamoto2004, it seems use $\Omega_s = -\frac{q_s B_0}{m_s}$, not the standard $\Omega_s = \frac{q_s B_0}{m_s}$?! This will affect the sign before the term $n\Omega_s$ and α. Is this why Ying TANG and Jin-song ZHAO at PMO found (2017-05) the polarization of $\boldsymbol{E}$ and $\boldsymbol{B}$ in the PDRK solution are opposite to usual case?

1.3 The linear transformation

To seek an equivalent linear system, the Maxwells equations

$$\partial_t \boldsymbol{E} = c^2 \nabla \times \boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{4a}$$
$$\partial_t \boldsymbol{B} = -\nabla \times \boldsymbol{E}, \tag{4b}$$

do not need to be changed. We only need to seek a new linear system for $\boldsymbol{J} = \overleftrightarrow{\sigma} \cdot \boldsymbol{E}$. It is easy to find that after J-pole expansion, the relations between $\boldsymbol{J}$ and $\boldsymbol{E}$ has the following form

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = \begin{pmatrix} a_{11} + \sum_{snjm} \frac{b_{snjm11}}{\omega - c_{snjm11}} & a_{12} + \sum_{snjm} \frac{b_{snjm12}}{\omega - c_{snjm12}} & a_{13} + \sum_{snjm} \frac{b_{snjm13}}{\omega - c_{snjm13}} \\ a_{21} + \sum_{snjm} \frac{b_{snjm21}}{\omega - c_{snjm21}} & a_{22} + \sum_{snjm} \frac{b_{snjm22}}{\omega - c_{snjm22}} & a_{23} + \sum_{snjm} \frac{b_{snjm23}}{\omega - c_{snjm23}} \\ a_{31} + \sum_{snjm} \frac{b_{snjm31}}{\omega - c_{snjm31}} & a_{32} + \sum_{snjm} \frac{b_{snjm32}}{\omega - c_{snjm32}} & a_{33} + \sum_{snjm} \frac{b_{snjm33}}{\omega - c_{snjm33}} + d_{33}\omega \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{5}$$

Fortunately, noting the relations in Z function ($\sum_j b_j = -1$, $\sum_j b_j c_j = 0$ and $\sum_j b_j c_j^2 = -1/2$) and in Bessel functions $[\sum_{n=-\infty}^{\infty} I_n(b) = e^b$, $\sum_{n=-\infty}^{\infty} n I_n(b) = 0$, $\sum_{n=-\infty}^{\infty} n^2 I_n(b) = be^b]$, we find that $a_{ij} = 0$ $(i, j = 1, 2, 3)$ and $d_{33} = 0$. Eq.(5) can be changed further to

$$\begin{pmatrix} J_x \\ J_y \\ J_z \end{pmatrix} = -i\epsilon_0 \begin{pmatrix} \frac{b_{11}}{\omega} + \sum_{snj} \frac{b_{snj11}}{\omega - c_{snj}} & \frac{b_{12}}{\omega} + \sum_{snj} \frac{b_{snj12}}{\omega - c_{snj}} & \frac{b_{13}}{\omega} + \sum_{snj} \frac{b_{snj13}}{\omega - c_{snj}} \\ \frac{b_{21}}{\omega} + \sum_{snj} \frac{b_{snj21}}{\omega - c_{snj}} & \frac{b_{22}}{\omega} + \sum_{snj} \frac{b_{snj22}}{\omega - c_{snj}} & \frac{b_{23}}{\omega} + \sum_{snj} \frac{b_{snj23}}{\omega - c_{snj}} \\ \frac{b_{31}}{\omega} + \sum_{snj} \frac{b_{snj31}}{\omega - c_{snj}} & \frac{b_{32}}{\omega} + \sum_{snj} \frac{b_{snj32}}{\omega - c_{snj}} & \frac{b_{33}}{\omega} + \sum_{snj} \frac{b_{snj33}}{\omega - c_{snj}} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \\ E_z \end{pmatrix}. \tag{6}$$

Combining Eqs. (4) and (6), the equivalent linear system for (1) can be obtained as[3]

$$\left\{ \begin{array}{rcl} \omega v_{snjx} &=& c_{snj} v_{snjx} + b_{snj11} E_x + b_{snj12} E_y + b_{snj13} E_z, \\ \omega j_x &=& b_{11} E_x + b_{12} E_y + b_{13} E_z, \\ i J_x/\epsilon_0 &=& j_x + \sum_{snj} v_{snjx}, \\ \omega v_{snjy} &=& c_{snj} v_{snjy} + b_{snj21} E_x + b_{snj22} E_y + b_{snj23} E_z, \\ \omega j_y &=& b_{21} E_x + b_{22} E_y + b_{23} E_z, \\ i J_y/\epsilon_0 &=& j_y + \sum_{snj} v_{snjy}, \\ \omega v_{snjz} &=& c_{snj} v_{snjz} + b_{snj31} E_x + b_{snj32} E_y + b_{snj33} E_z, \\ \omega j_z &=& b_{31} E_x + b_{32} E_y + b_{33} E_z, \\ i J_z/\epsilon_0 &=& j_z + \sum_{snj} v_{snjz}, \\ \omega E_x &=& +c^2 k_z B_y - i J_x/\epsilon_0, \\ \omega E_y &=& -c^2 k_z B_x + c^2 k_x B_z - i J_y/\epsilon_0, \\ \omega E_z &=& -c^2 k_x B_y - i J_z/\epsilon_0, \\ \omega B_x &=& -k_z E_y, \\ \omega B_y &=& +k_z E_x - k_x E_z, \\ \omega B_z &=& +k_x E_y, \end{array} \right. \tag{7}$$

which yields a sparse matrix eigenvalue problem. Again, the symbols v_{snjx}, $j_{x,y,z}$ and $J_{x,y,z}$ used here do not have direct physical meanings but are analogy to the perturbed velocity and current density in the fluid derivations of plasma waves. The elements of the eigenvector $(E_x, E_y, E_z, B_x, B_y, B_z)$

[3] The signs of $\boldsymbol{E}$ and $\boldsymbol{B}$ terms in the right hand side of the below Maxwell equation ωE and ωB has been opposite in the original paper [Xie2016], but the signs in the pdrk-em3d code are correct.

still represent the original electric and magnetic fields. Thus, the polarization of the solutions can also be obtained in a straightforward manner. The dimension of the matrix is $NN = 3 \times (SNJ + 1) + 6 = 3 \times [S \times (2 \times N + 1) \times J + 1] + 6$. The coefficients in original paper [Xie2014] are

$$\left\{\begin{array}{ccc}
b_{snj11} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj}) n^2 \Gamma_n / b_s, \\
b_{11} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) n^2 \Gamma_n / b_s, \\
b_{snj12} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj}) in\Gamma_n', \\
b_{12} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) in\Gamma_n', \\
b_{snj21} = -b_{snj12} & , & b_{21} = -b_{12}, \\
b_{snj22} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n'), \\
b_{22} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n'), \\
b_{snj13} & = & \omega_{ps}^2 b_j[c_j/\lambda_{Ts} - n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})] n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\
b_{13} & = & \sum_{snj} \omega_{ps}^2 b_j [n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})] n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\
b_{snj31} = b_{snj13} & , & b_{31} = b_{13}, \\
b_{snj23} & = & -i\omega_{ps}^2 b_j[c_j/\lambda_{Ts} - n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\
b_{23} & = & -i\sum_{snj} \omega_{ps}^2 b_j [n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\
b_{snj32} = -b_{snj23} & , & b_{32} = -b_{23}, \\
b_{snj33} & = & \omega_{ps}^2 b_j[(v_{s0}/v_{tzs} + c_j)c_j/\lambda_{Ts} - n\omega_{cs} b_{j0}((1 + n\omega_{cs}/c_{snj})/v_{tzs}^2)/k_z] 2\lambda_{Ts}\Gamma_n, \\
b_{33} & = & \sum_{snj} \omega_{ps}^2 b_j [n^2 b_{j0}/(c_{snj} v_{tzs}^2 k_z)] 2\lambda_{Ts}\Gamma_n, \\
c_{snj} & = & k_z c_j v_{tzs} + k_z v_{s0} - n\omega_{cs},
\end{array}\right. \tag{8}$$

the correction one should be

$$\left\{\begin{array}{ccc}
b_{snj11} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj}) n^2 \Gamma_n / b_s, \\
b_{11} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) n^2 \Gamma_n / b_s, \\
b_{snj12} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj}) in\Gamma_n', \\
b_{12} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) in\Gamma_n', \\
b_{snj21} = -b_{snj12} & , & b_{21} = -b_{12}, \\
b_{snj22} & = & \omega_{ps}^2 b_j(1 - k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n'), \\
b_{22} & = & \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n'), \\
b_{snj13} & = & \omega_{ps}^2 b_j[c_j/\lambda_{Ts} + n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})] n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\
b_{13} & = & -\sum_{snj} \omega_{ps}^2 b_j [n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})] n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\
b_{snj31} = b_{snj13} & , & b_{31} = b_{13}, \\
b_{snj23} & = & -i\omega_{ps}^2 b_j[c_j/\lambda_{Ts} + n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\
b_{23} & = & +i\sum_{snj} \omega_{ps}^2 b_j [n\omega_{cs} b_{j0}/(c_{snj} v_{tzs})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\
b_{snj32} = -b_{snj23} & , & b_{32} = -b_{23}, \\
b_{snj33} & = & \omega_{ps}^2 b_j[(c_j/\lambda_{Ts} + b_{j0} n\omega_{cs}/(c_{snj} v_{tzs})](v_{s0}/v_{tzs} + c_j) 2\lambda_{Ts}\Gamma_n, \\
b_{33} & = & \sum_{snj} \omega_{ps}^2 b_j [n^2 \omega_{cs}^2 b_{j0}/(c_{snj} v_{tzs}^2 k_z)] 2\lambda_{Ts}\Gamma_n, \\
c_{snj} & = & k_z c_j v_{tzs} + k_z v_{s0} + n\omega_{cs},
\end{array}\right. \tag{9}$$

where $b_{j0} = v_{s0} + (1 - 1/\lambda_{Ts}) c_j v_{tzs}$. To avoid the $\sqrt{2}$ in/from ζ_n and η_n, we have redefined $v_{tzs} = \sqrt{\frac{k_B T_{sz}}{m_s}} = \sqrt{2} v_{zts} = \sqrt{2} v_{Tz}$ in the above Eq.(9) and the PDRK code. Note that we have used $b_{snj33} = \omega_{ps}^2 b_j[(c_j/\lambda_{Ts} + b_{j0} n\omega_{cs}/(c_{snj} v_{tzs})](v_{s0}/v_{tzs} + c_j) 2\lambda_{Ts}\Gamma_n$, instead of the original $b_{snj33} = \omega_{ps}^2 b_j[(v_{s0}/v_{tzs} + c_j)c_j/\lambda_{Ts} + n\omega_{cs} b_{j0}((1 - n\omega_{cs}/c_{snj})/v_{tzs}^2)/k_z] 2\lambda_{Ts}\Gamma_n$, though they are equivalent (see appendix).

If $a_{ij} \neq 0$, then the equivalent linear transformation is still straightforward. If $d_{33} \neq 0$, then the equivalent linear transformation will be more complicated. For our purposes, we do not need to discuss these cases.

We note that the major bug is a missed ω_{cs}^2 on b_{33} term. For $k_\perp \to 0$ or $T_s \to 0$, we have $b_s \to 0$, and $\Gamma_0(b_s) \to 1$, $\Gamma_n(b_s) \to 0$ for $n \neq 0$, thus $b_{33} = \sum_{snj} \omega_{ps}^2 b_j [n^2 \omega_{cs}^2 b_{j0}/(c_{snj} v_{tzs}^2 k_z)] 2\lambda_{Ts}\Gamma_n = 0$. This is why the [Xie2014] version pdrk-em3d is correct for parallel propagation ($k_\parallel \neq 0$, $k_\perp = 0$) mode and cold plasma ($T_s \to 0$) mode.

2 Benchmark

Based on above new derivations, especially fixed a missed ω_{cs}^2 bug on b_{33} term, the mirror mode case can agree with WHAMP now (see Fig.1), whereas the old version have bugs and disagree (see Fig.2). The benchmark case is from Gary1993 book p131, Fig.7.4, with $\theta = 71°$, $\beta_{//p} = \beta_{//e} = 1$, $T_{p,\perp}/T_{p,//} = 2$ and $T_{e,\perp}/T_{e,//} = 1$. And the WHAMP data is provided (2018-04-14) by Richard Denton.

The input data in pdrk.in are

```
qs          ms           ns           Tzs            Tps           vs0
-1          1            1.e6         24840.         24840.        0.0
 1          1836         1.e6         24840.         49680.        0.0
```

and magnetic field B0=100.0E-9 T.

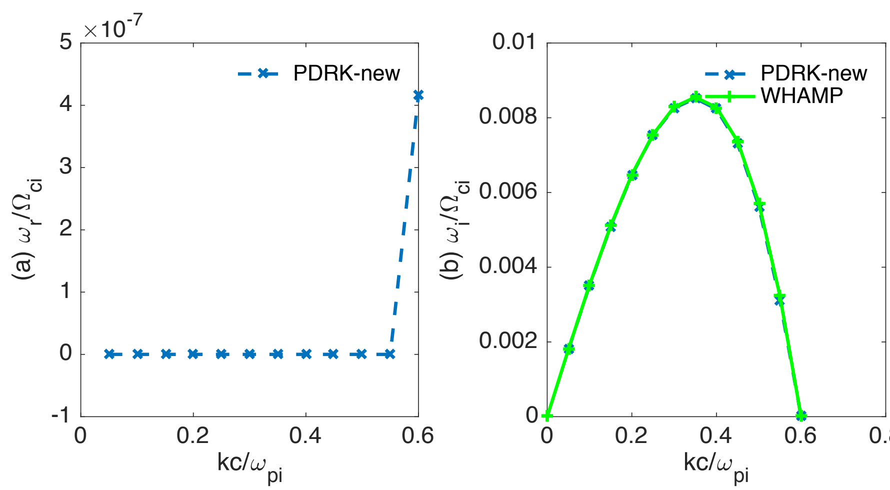

Figure 1: The new version of pdrk-em3d (2018-10-01 10:22) agree with WHAMP for the mirror mode.

2018-10-01 14:36 There seems still exist slight differences for benchmark [Li2000] with $\theta \neq 0$. [Li2000] Xing Li and Shadia Rifai Habbal, Electron kinetic firehose instability, Journal of Geophysical Research: Space Physics, 105 (A12), 27377, 2000. [2018-10-03 08:20 update] However, PDRK agrees well with Denton's WHAMP result for Li2000 Fig1d ($\theta = 22°$) parameters ($v_A/c = 0.001$, $\beta_{\parallel e} = 5$, $\beta_{\parallel p} = 1$, $T_{\perp e}/T_{\parallel e} = 0.5$), we used $B = 100nT$, $n_p = n_e = 52.9cm^{-3}$, $T_{\parallel p} = 0.470keV$,

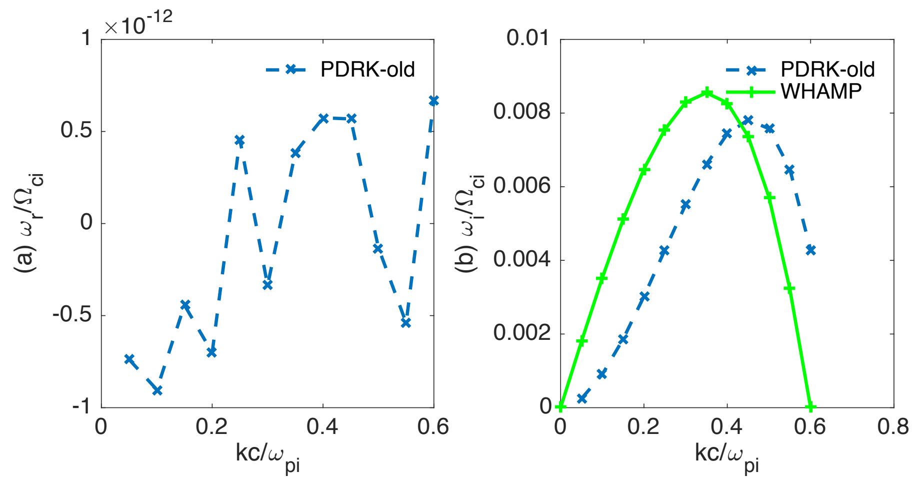

Figure 2: The old version of pdrk-em3d [Xie2014] disagree with WHAMP for the mirror mode when $k_\perp \rho_{cs} \neq 0$, due to the bugs in the original code.

$T_{\parallel e} = 2.35keV$, $T_{\perp e}/T_{\parallel e} = 0.5$. The slight disagreement with Li2000 is probably due to the difference of magnetic field B and density n (not given in the paper), though v_A/c is the same.

2018-10-01 22:04 Benchmark Gary1993 book, Fig.7.1 and Fig.7.2, the $\theta = 0$, $T_\perp \neq T_\parallel$ modes agree. The Fig.8.2 and Fig.8.3, $\theta = 0$ beam modes also agree or at least very similar. 2018-10-02 12:56 Fig.8.6 $\theta \neq 0$ beam modes also agree.

For summary, we think this new version of pdrk-em3d have pasted the essential benchmarks and can be trust now.

3 More words

The good aspects of PDRK are that: It does not need initial guess of root finding and can gives all the physical solutions (except strong damped solutions) quickly.

One can also use the sparse matrix eigen matrix function 'eigs()' instead of 'eig()' in MATLAB if you only need one or several solutions around an initial guess value, which could be similar to other iterative root finding solver (say WHAMP). For large Bessel function summation number N (e.g., $N > 10$), the matrix dimension are large, and one can use 'eigs()', which can be much faster than 'eig()'. Thus, PDRK-EM3D is superior to WHAMP at all aspects.

The major drawback of pdrk is the extraneous roots. Strategies to filter out those extraneous roots are required.

Possible to do: Rewrite a user friendly version of PDRK.

4 Acknowledgement

I should thank many people to draw my attention that the PDRK paper/code may not be inconsistent, especially the careful benchmark data provided (2018-06) by Prof. Richard Denton.

A The electromagnetic dispersion relation

A.1 Basic idea

Firstly [Gurnett2005 sec.9.3]

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E})+\frac{\omega^2}{c^2}\boldsymbol{K}\cdot\boldsymbol{E}=0, \tag{10}$$

where $\boldsymbol{E}$ is the electric field of the wave and $\boldsymbol{K}$ is the dielectric tensor. And the current density

$$\boldsymbol{J}=\boldsymbol{\sigma}\cdot\boldsymbol{E}. \tag{11}$$

Once the conductivity tensor $\boldsymbol{\sigma}$ is known, the dielectric tensor can be computed using $\boldsymbol{K}=\boldsymbol{I}-\boldsymbol{\sigma}/(i\omega\epsilon_0)$. Compared with Eq.(1), we find that for EM3D dispersion relation

$$\boldsymbol{\sigma}=-i\epsilon_0\sum_s\frac{\omega_{ps}^2}{\omega}\Big[\sum_n\big\{\zeta_0Z(\zeta_n)-(1-\frac{1}{\lambda_T})[1+\zeta_nZ(\zeta_n)]\big\}\boldsymbol{X}_n+2\eta_0^2\lambda_T\boldsymbol{L}\Big]. \tag{12}$$

Combine the above equation and

$$Z(\zeta)\simeq Z_J(\zeta)=\sum_{j=1}^{J}\frac{b_j}{\zeta-c_j}, \tag{13}$$

we can obtain Eq.(5) and the final transform matrix. Here, b_j and c_j are constants for given J, as given in [Xie2016].

A.2 More details

Note

$$\frac{1}{\omega}\frac{b}{\omega-c}=\frac{b}{c}\Big(\frac{1}{\omega-c}-\frac{1}{\omega}\Big),$$

we have

$$\begin{aligned}
Y &\equiv \zeta_0Z(\zeta_n)-(1-\frac{1}{\lambda_T})[1+\zeta_nZ(\zeta_n)]\\
&= \sum_{j=1}^{J}\frac{\zeta_0b_j}{\zeta_n-c_j}-(1-\frac{1}{\lambda_T})\Big[1+\sum_{j=1}^{J}\frac{\zeta_nb_j}{\zeta_n-c_j}\Big]\\
&= \sum_{j=1}^{J}b_j\Big[1+\frac{c_jk_zv_{tzs}+n\Omega_s}{\omega-c_{snj}}\Big]-(1-\frac{1}{\lambda_T})\Big[1+\sum b_j+\sum_{j=1}^{J}\frac{b_jc_j}{\zeta_n-c_j}\Big]\\
&= -1+\sum_{j=1}^{J}\frac{b_jc_jk_zv_{tzs}+b_jn\Omega_s}{\omega-c_{snj}}-(1-\frac{1}{\lambda_T})\sum_{j=1}^{J}\frac{b_jc_jk_zv_{tzs}}{\omega-c_{snj}}\\
&= -1+\sum_{j=1}^{J}\frac{b_j(c_jk_zv_{tzs}/\lambda_T+n\Omega_s)}{\omega-c_{snj}},
\end{aligned}$$

where we have defined $c_{snj} = k_z v_{s0} + n\Omega_s + k_z v_{tzs} c_j$ and used $\sum_{j=1}^J b_j = -1$, and thus

$$
\begin{aligned}
A &\equiv \frac{Y}{\omega} = -\frac{1}{\omega} + \frac{1}{\omega}\sum_{j=1}^J \frac{b_j(c_j k_z v_{tzs}/\lambda_T + n\Omega_s)}{\omega - c_{snj}} \\
&= -\frac{1}{\omega} + \sum_{j=1}^J \frac{b_j(c_j k_z v_{tzs}/\lambda_T + n\Omega_s)/c_{snj}}{\omega - c_{snj}} - \sum_{j=1}^J \frac{b_j(c_j k_z v_{tzs}/\lambda_T + n\Omega_s)/c_{snj}}{\omega} \\
&= \sum_{j=1}^J \frac{b_j(c_j k_z v_{tzs}/\lambda_T + n\Omega_s)/c_{snj}}{\omega - c_{snj}} + \sum_{j=1}^J \frac{b_j[1 - (c_j k_z v_{tzs}/\lambda_T + n\Omega_s)/c_{snj}]}{\omega} \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big[\frac{(c_j k_z v_{tzs}/\lambda_T + n\Omega_s)}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big] = \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big(\frac{c_{snj} - k_z b_{j0}}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big),
\end{aligned}
$$

where we have defined $b_{j0} = v_{s0} + (1 - 1/\lambda_T)c_j v_{tzs}$, and note that $c_j k_z v_{tzs}/\lambda_T + n\Omega_s = c_{snj} - k_z b_{j0}$. And

$$
\begin{aligned}
\eta_n A &= \frac{\omega - n\Omega_s}{k_z v_{tzs}} \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big(\frac{c_{snj} - k_z b_{j0}}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big) \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj} k_z v_{tzs}}\Big[(c_{snj} - k_z b_{j0})\Big(1 + \frac{k_z v_{s0} + k_z v_{tzs} c_j}{\omega - c_{snj}}\Big) + k_z b_{j0} - \frac{k_z b_{j0} n\Omega_s}{\omega}\Big] \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj} k_z v_{tzs}}\Big[(c_{snj} - k_z b_{j0})\frac{(k_z v_{s0} + k_z v_{tzs} c_j)}{\omega - c_{snj}} - \frac{k_z b_{j0} n\Omega_s}{\omega} + (c_{snj} - k_z b_{j0}) + k_z b_{j0}\Big] \\
&= -\frac{1}{k_z v_{tzs}} + \sum_{j=1}^J \frac{b_j}{c_{snj} v_{tzs}}\Big[(c_{snj} - k_z b_{j0})\frac{(v_{s0} + v_{tzs} c_j)}{\omega - c_{snj}} - \frac{b_{j0} n\Omega_s}{\omega}\Big] \\
&= -\frac{1}{k_z v_{tzs}} + \sum_{j=1}^J \frac{b_j}{c_{snj} v_{tzs}}\Big(\frac{c_{snj} c_j v_{tzs}/\lambda_T + n\Omega_s b_{j0}}{\omega - c_{snj}} - \frac{b_{j0} n\Omega_s}{\omega}\Big),
\end{aligned}
$$

where we have used that $c_{snj}c_jv_{tzs}/\lambda_T + n\Omega_s b_{j0} = (k_zv_{s0} + n\Omega_s + k_zv_{tzs}c_j)c_jv_{tzs}/\lambda_T + n\Omega_s v_{s0} + n\Omega_s(1-1/\lambda_T)c_jv_{tzs} = (c_jk_zv_{tzs}/\lambda_T + n\Omega_s)(v_{s0}+v_{tzs}c_j) = (c_{snj}-k_zb_{j0})(v_{s0}+v_{tzs}c_j)$. And

$$\begin{aligned}
\eta_n^2 A &= \frac{\omega - n\Omega_s}{k_zv_{tzs}}\Big\{-\frac{1}{k_zv_{tzs}} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}v_{tzs}}\Big[(c_{snj}-k_zb_{j0})\frac{(v_{s0}+v_{tzs}c_j)}{\omega - c_{snj}} - \frac{b_{j0}n\Omega_s}{\omega}\Big]\Big\} \\
&= -\frac{\omega}{k_z^2v_{tzs}^2} + \frac{n\Omega_s}{k_z^2v_{tzs}^2} + (\omega - n\Omega_s)\sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{tzs}^2}\Big[(c_{snj}-k_zb_{j0})\frac{(v_{s0}+v_{tzs}c_j)}{\omega - c_{snj}} - \frac{b_{j0}n\Omega_s}{\omega}\Big] \\
&= -\frac{\omega}{k_z^2v_{tzs}^2} + \frac{n\Omega_s}{k_z^2v_{tzs}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{tzs}^2}\Big[(c_{snj}-k_zb_{j0})(v_{s0}+v_{tzs}c_j)\Big(1+\frac{k_zv_{s0}+k_zv_{tzs}c_j}{\omega - c_{snj}}\Big) \\
&\quad +\frac{b_{j0}n^2\Omega_s^2}{\omega} - b_{j0}n\Omega_s\Big] \\
&= -\frac{\omega}{k_z^2v_{tzs}^2} + \frac{n\Omega_s}{k_z^2v_{tzs}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{tzs}^2}\Big[(c_{snj}-k_zb_{j0})\frac{k_z(v_{s0}+v_{tzs}c_j)^2}{\omega - c_{snj}} + \frac{b_{j0}n^2\Omega_s^2}{\omega}\Big] + \sum_{j=1}^{J}\frac{b_jc_j}{k_zv_{tzs}} \\
&= -\frac{\omega}{k_z^2v_{tzs}^2} + \frac{n\Omega_s}{k_z^2v_{tzs}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{tzs}^2}\Big[(c_{snj}-k_zb_{j0})\frac{k_z(v_{s0}+v_{tzs}c_j)^2}{\omega - c_{snj}} + \frac{b_{j0}n^2\Omega_s^2}{\omega}\Big],
\end{aligned}$$

where we have noticed $(c_{snj}-k_zb_{j0})(v_{s0}+v_{tzs}c_j) - b_{j0}n\Omega_s = (c_jk_zv_{tzs}/\lambda_T + n\Omega_s)(v_{s0}+v_{tzs}c_j) - b_{j0}n\Omega_s = (v_{s0}c_jk_zv_{tzs}/\lambda_T + v_{s0}n\Omega_s + v_{tzs}c_jc_jk_zv_{tzs}/\lambda_T + v_{tzs}c_jn\Omega_s) - n\Omega_sv_{s0} - n\Omega_s(1-1/\lambda_T)c_jv_{tzs} = (v_{s0}c_jk_zv_{tzs}/\lambda_T + v_{tzs}c_jc_jk_zv_{tzs}/\lambda_T + n\Omega_sc_jv_{tzs}/\lambda_T) = c_{snj}c_jv_{tzs}/\lambda_T$, and used $\sum_{j=1}^{J} b_jc_j = 0$.

Use Eq.(3) and compare with Eq.(5):

- A has no constant term, $\Rightarrow a_{11} = a_{12} = a_{21} = a_{22} = 0$.
- $\sum_{n=-\infty}^{\infty} n\Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} nI_n = e^{-b}\sum_{n=1}^{\infty} n(I_n - I_{-n}) = 0$, and η_nA has only constant term $-\frac{1}{k_zv_{tzs}}$, $\Rightarrow a_{13} = a_{31} = 0$.
- $\sum_{n=-\infty}^{\infty} \Gamma'_n = e^{-b}\sum_{n=-\infty}^{\infty}(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = 0$, and η_nA has only constant term $-\frac{1}{k_zv_{tzs}}$, $\Rightarrow$ $a_{23} = a_{32} = 0$.
- $\sum_{n=-\infty}^{\infty} \Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} I_n = 1$. Thus using η_n^2A, the first two terms of σ_{33}, i.e., terms ω^1 and ω^0 are, $-i\epsilon_0\sum_s\omega_{ps}^2\Big[\sum_n 2\lambda_T\Gamma_n\Big(-\frac{\omega}{k_z^2v_{tzs}^2} + \frac{n\Omega_s}{k_z^2v_{tzs}^2}\Big) + \frac{2\eta_0^2\lambda_T}{\omega}\Big] = -i\epsilon_0\sum_s\omega_{ps}^2\Big[-2\lambda_T\frac{\omega}{k_z^2v_{tzs}^2} + \frac{2\omega\lambda_T}{k_z^2v_{tzs}^2}\Big] = 0$, $\Rightarrow a_{33} = 0$ and $d_{33} = 0$.

Not used yet: $\sum_{n=-\infty}^{\infty} n\Gamma'_n = e^{-b}\sum_{n=-\infty}^{\infty} n(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = e^{-b}\sum_{n=1}^{\infty} n(\frac{I_{n+1}+I_{n-1}-I_{-n+1}-I_{-n-1}}{2} - I_n + I_{-n}) = 0$.

After the above steps, we can obtain Eq.(6) and corresponding coefficients in Eq.(9). For examples

- $b_{11} = \sum_s\omega_{ps}^2\sum_n\frac{n^2\Gamma_n}{b_s}\sum_{j=1}^{J}\frac{b_jk_zb_{j0}}{c_{snj}} = \sum_{snj}\omega_{ps}^2b_j(k_zb_{j0}/c_{snj})n^2\Gamma_n/b_s$.
- $b_{snj11} = \omega_{ps}^2\frac{n^2\Gamma_n}{b_s}\frac{b_j(c_{snj}-k_zb_{j0})}{c_{snj}} = \omega_{ps}^2b_j(1-k_zb_{j0}/c_{snj})n^2\Gamma_n/b_s$.

- $b_{12} = \sum_s \omega_{ps}^2 \sum_n in\Gamma'_n \sum_{j=1}^J \frac{b_j k_z b_{j0}}{c_{snj}} = \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) in\Gamma'_n$.
- $b_{snj12} = \omega_{ps}^2 in\Gamma'_n \frac{b_j(c_{snj}-k_z b_{j0})}{c_{snj}} = \omega_{ps}^2 b_j (1 - k_z b_{j0}/c_{snj}) in\Gamma'_n$.
- $b_{snj21} = -b_{snj12}$, $b_{21} = -b_{12}$.
- $b_{22} = \sum_s \omega_{ps}^2 \sum_n (n^2\Gamma_n/b_s - 2b_s\Gamma'_n) \sum_{j=1}^J \frac{b_j k_z b_{j0}}{c_{snj}} = \sum_{snj} \omega_{ps}^2 b_j (k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma'_n)$.
- $b_{snj22} = \omega_{ps}^2 (n^2\Gamma_n/b_s - 2b_s\Gamma'_n) \frac{b_j(c_{snj}-k_z b_{j0})}{c_{snj}} = \omega_{ps}^2 b_j (1 - k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma'_n)$.
- $b_{13} = -\sum_s \omega_{ps}^2 \sum_n n\Gamma_n (\sqrt{2\lambda_{Ts}}/\alpha_s) \sum_{j=1}^J \frac{b_j b_{j0} n\Omega_s}{c_{snj} v_{tzs}} = -\sum_{snj} \omega_{ps}^2 b_j [b_{j0} n\Omega_s/(c_{snj} v_{tzs})] \sqrt{2\lambda_{Ts}} n\Gamma_n/\alpha_s$.
- $b_{snj13} = \omega_{ps}^2 n\Gamma_n (\sqrt{2\lambda_{Ts}}/\alpha_s) \frac{b_j}{c_{snj} v_{tzs}} (c_{snj} c_j v_{tzs}/\lambda_{Ts} + n\Omega_s b_{j0}) = \omega_{ps}^2 b_j [c_j/\lambda_{Ts} + n\Omega_s b_{j0}/(c_{snj} v_{tzs})]$ $\sqrt{2\lambda_{Ts}} n\Gamma_n/\alpha_s$.
- $b_{snj31} = b_{snj13}$, $b_{31} = b_{13}$.
- $b_{23} = i\sum_s \omega_{ps}^2 \sum_n \Gamma'_n (\sqrt{2\lambda_{Ts}}\alpha_s) \sum_{j=1}^J \frac{b_j b_{j0} n\Omega_s}{c_{snj} v_{tzs}} = i\sum_{snj} \omega_{ps}^2 b_j [b_{j0} n\Omega_s/(c_{snj} v_{tzs})] \sqrt{2\lambda_{Ts}} \Gamma'_n \alpha_s$.
- $b_{snj23} = -i\omega_{ps}^2 \Gamma'_n (\sqrt{2\lambda_{Ts}}\alpha_s) \frac{b_j}{c_{snj} v_{tzs}} (c_{snj} c_j v_{tzs}/\lambda_{Ts} + n\Omega_s b_{j0}) = -i\omega_{ps}^2 b_j [c_j/\lambda_{Ts} + n\Omega_s b_{j0}/(c_{snj} v_{tzs})]$ $\sqrt{2\lambda_{Ts}} \Gamma'_n \alpha_s$.
- $b_{snj32} = -b_{snj23}$, $b_{32} = -b_{23}$.
- $b_{33} = \sum_s \omega_{ps}^2 \sum_n 2\lambda_{Ts}\Gamma_n \sum_{j=1}^J \frac{b_j b_{j0} n^2\Omega_s^2}{c_{snj} k_z v_{tzs}^2} = \sum_{snj} \omega_{ps}^2 b_j [b_{j0} n^2\Omega_s^2/(c_{snj} k_z v_{tzs}^2)] 2\lambda_{Ts}\Gamma_n$.
- $b_{snj33} = \omega_{ps}^2 2\lambda_{Ts}\Gamma_n \frac{b_j(c_{snj}-k_z b_{j0})(v_{s0}+v_{tzs}c_j)^2}{c_{snj} v_{tzs}^2} = \omega_{ps}^2 b_j \frac{(c_{snj} c_j v_{tzs}/\lambda_T + b_{j0} n\Omega_s)(v_{s0}+v_{tzs}c_j)}{c_{snj} v_{tzs}^2} 2\lambda_{Ts}\Gamma_n$
  $= \omega_{ps}^2 b_j [(c_j/\lambda_{Ts} + b_{j0} n\Omega_s/(c_{snj} v_{tzs})](v_{s0}/v_{tzs} + c_j) 2\lambda_{Ts}\Gamma_n$
  $= \omega_{ps}^2 b_j [(v_{s0}/v_{tzs} + c_j) c_j/\lambda_{Ts} + n\Omega_s b_{j0} (v_{s0}/v_{tzs} + c_j)/(c_{snj} v_{tzs})] 2\lambda_{Ts}\Gamma_n$
  $= \omega_{ps}^2 b_j [(v_{s0}/v_{tzs} + c_j) c_j/\lambda_{Ts} + n\Omega_s b_{j0} (1 - n\Omega_s/c_{snj})/(k_z v_{tzs}^2)] 2\lambda_{Ts}\Gamma_n$.

B Refresh some memory

Checked the original derivations (2014 summer) of pdrk-em3d project, it starts (2014-05-31 to 14-06-05, and updated/benchmarked during 14-08-15 to 14-08-28) from the $T_\perp = T_\parallel$ and $v_{s0} = 0$ dispersion relation in p400 of Xiwei Hu's 2006 plasma theory book (in Chinese) and then use the drift bi-Maxwellian version in Miyamoto2004 p209. And some notations come from the ES1D and ES3D version. This is the reason of inconsistent notations/typos/bugs in PDRK-EM3D document and code.

Last update: Wednesday 3$^{\rm rd}$ October, 2018 13:04.