Generalize PDRK-EM3D Dispersion Relation Solver from Drift bi-Maxwellian Distribution to with Loss-cone

Hua-sheng XIE, huashengxie@gmail.com, FRI-ENN

Saturday 27th October, 2018

Start: 2018-10-06 17:42

The first version of PDRK-EM3D [Xie2016] support drift bi-Maxwellian distribution $f_{s0} = n_{s0} f_\perp(v_\perp) f_z(v_z)$, with $f_z = (\frac{m_s}{2\pi k_B T_{sz}})^{1/2} \exp[-\frac{m_s(v_\parallel - v_{ds})^2}{2k_B T_{sz}}]$ and $f_\perp = \frac{m_s}{2\pi k_B T_{s\perp}} \exp[-\frac{m_s v_\perp^2}{2k_B T_{s\perp}}]$. In this new version, we update it to also support the loss-cone distribution and thus can handle all the same cases as in WHAMP [Ronnmark1982], with

$$
\begin{aligned}
f_{0j}(v_\parallel, v_\perp) \quad &= \quad \frac{1}{(\pi^{1/2} v_j)^3} \exp\Big[-(\frac{v_\parallel}{v_j} - v_{dj})^2\Big]\Big\{\frac{\Delta_j}{\alpha_{1j}} \exp\Big(-\frac{v_\perp^2}{\alpha_{1j} v_j^2}\Big) \\
&\quad + \frac{1-\Delta_j}{\alpha_{1j} - \alpha_{2j}}\Big[\exp\Big(-\frac{v_\perp^2}{\alpha_{1j} v_j^2}\Big) - \exp\Big(-\frac{v_\perp^2}{\alpha_{2j} v_j^2}\Big)\Big]\Big\}
\end{aligned}
$$

Here, v_j is the thermal velocity of a component with temperature $T_j = \frac{1}{2} m_j v_j^2$, and v_{dj} is a normalized drift velocity along the magnetic field. The parameters Δ_j, α_{1j} and α_{2j} determine the depth and size of the loss-cone and the temperature anisotrophy.

The other major updates are resolve the artificial solutions problem and provide a method to separate different dispersion surface automatically.

Refs:

[Xie2016] Huasheng Xie and Yong Xiao, PDRK: A General Kinetic Dispersion Relation Solver for Magnetized Plasma, Plasma Science and Technology, 18, 2, 97 (2016). http://hsxie.me/codes/pdrk/

[Miyamoto2004] Kenro Miyamoto, Plasma Physics and Controlled Nuclear Fusion, Springer, 2004.

[Ronnmark1982] Kjell Ronnmark, WHAMP - Waves in Homogeneous Anisotropic Multicomponent Magnetized Plasma, KGI Report, No. 179, 1982.

1 PDRK Equation

1.1 Equilibrium distribution function

We combine the PDRK and WHAMP notation of the drift bi-Maxwellian loss cone equilibrium distribution function $F_{s0}(v_\parallel, v_\perp) = n_{s0} f_{s0}(v_\parallel, v_\perp)$, with

$$
\begin{aligned}
& f_{s0}(v_\parallel, v_\perp) = f_\perp(v_\perp) f_z(v_\parallel) \qquad (1) \\
= \; & \frac{1}{\pi^{3/2} v_{zts} v_{\perp ts}^2} \exp\Big[-\frac{(v_\parallel - v_{ds})^2}{v_{zts}^2}\Big]\Big\{\Delta_s \exp\Big(-\frac{v_\perp^2}{v_{\perp ts}^2}\Big) + \frac{1-\Delta_s}{1-\alpha_s}\Big[\exp\Big(-\frac{v_\perp^2}{v_{\perp ts}^2}\Big) - \exp\Big(-\frac{v_\perp^2}{\alpha_s v_{\perp ts}^2}\Big)\Big]\Big\},
\end{aligned}
$$

i.e.,

$$f_z(v_\parallel) = \frac{1}{\pi^{1/2} v_{zts}} \exp\Big[-\frac{(v_\parallel - v_{ds})^2}{v_{zts}^2}\Big], \tag{2}$$

and

$$f_\perp(v_\perp) = \frac{1}{\pi v_{\perp ts}^2}\Big\{\Delta_s \exp\Big(-\frac{v_\perp^2}{v_{\perp ts}^2}\Big) + \frac{1-\Delta_s}{1-\alpha_s}\Big[\exp\Big(-\frac{v_\perp^2}{v_{\perp ts}^2}\Big) - \exp\Big(-\frac{v_\perp^2}{\alpha_s v_{\perp ts}^2}\Big)\Big]\Big\}, \tag{3}$$

where v_{ds} is the parallel drift velocity, the v_{zts} and $v_{\perp ts}$ are the parallel and perpendicular thermal velocities and corresponding temperatures are $T_{zs} = \frac{1}{2} k_B m_s v_{zts}^2$ and $T_{\perp s} = \frac{1}{2} k_B m_s v_{\perp ts}^2$. We define the temperature anisotropic $\lambda_{Ts} = T_{zs}/T_{\perp s}$. The parameters Δ_s and α_s determine the depth and size of the loss-cone. Here, $\Delta \in [0, 1]$, for max loss cone and no loss cone. If $\Delta_s = 1$ or $\alpha_s = 1$, the above equation reduced to drift bi-Maxwellian distribution.

We can separate $f_{s0\perp}$ to two sub-distributions $f_{s0\perp} = \Big(\frac{1-\alpha_s\Delta_s}{1-\alpha_s}\Big) f_{s0\perp a} + \Big(\frac{-\alpha_s+\alpha_s\Delta_s}{1-\alpha_s}\Big) f_{s0\perp b}$, with

$$f_{s0\perp a}(v_\perp) = \frac{1}{\pi v_{\perp ts}^2} \exp\Big(-\frac{v_\perp^2}{v_{\perp ts}^2}\Big), \tag{4}$$

and

$$f_{s0\perp b}(v_\perp) = \frac{1}{\pi \alpha_s v_{\perp ts}^2} \exp\Big(-\frac{v_\perp^2}{\alpha_s v_{\perp ts}^2}\Big). \tag{5}$$

And thus f_{s0} can be treated as two bi-Maxwellian distributions $f_{s0} = \Big(\frac{1-\alpha_s\Delta_s}{1-\alpha_s}\Big) f_{s0a} + \Big(\frac{-\alpha_s+\alpha_s\Delta_s}{1-\alpha_s}\Big) f_{s0b}$, with $\int f_{s0a} d\boldsymbol{v} = \int f_{s0b} d\boldsymbol{v} = \int f_{s0} d\boldsymbol{v} = 1$.

Note: Two perpendicular temperatures, and thus requires two b_s and a_s.

1.2 Notations

Note the definition of v_{ts}, i.e., $v_{ts} = \sqrt{\frac{2k_BT_s}{m_s}}$, not $v_{ts} = \sqrt{\frac{k_BT_s}{m_s}}$. Other notations: $\omega_{ps}^2 = \frac{n_{s0}q_s^2}{\epsilon_0 m_s}$, $\Omega_s = \frac{q_s B_0}{m_s}$, $\lambda_{Ds}^2 = \frac{\epsilon_0 k_B T_{zs}}{n_{s0} q_s^2}$, $\zeta_{sn} = \frac{\omega - k_z v_{ds} - n\Omega_s}{k_z v_{zts}}$, $\eta_{sn} = \frac{\omega - n\Omega_s}{k_z v_{zts}}$, $\omega_{sn} = \omega - k_z v_{ds} - n\Omega_s$, $a_s = k_\perp \rho_{cs}$, $b_s = k_\perp^2 \rho_{cs}^2$, $\rho_{cs} = \sqrt{\frac{k_B T_{s\perp}}{m_s}} \frac{1}{\Omega_s} = \frac{v_{\perp ts}}{\sqrt{2}\Omega_s}$, $\Gamma_n(b) = I_n(b) e^{-b}$, I_n is the modified Bessel function.

Note carefully that two perpendicular temperatures relevant terms, we have labelled them by a and b: $T_{\perp sa,b}$, $v_{\perp tsa,b}$, $\lambda_{Tsa,b}$, $b_{sa,b}$, $a_{sa,b}$ and $\rho_{csa,b}$.

1.3 Dispersion relation

1.3.1 Start equation

The background magnetic field is assumed to be $\boldsymbol{B}_0 = (0, 0, B_0)$, and the wave vector $\boldsymbol{k} = (k_x, 0, k_z) = (k\sin\theta, 0, k\cos\theta)$, which gives $k_\perp = k_x$ and $k_\parallel = k_z$. In the absence of external sources, the electric field $\boldsymbol{E}(\omega, \boldsymbol{k})$ of a wave with frequency ω and wave vector $\boldsymbol{k}$ satisfies a wave equation

$$\boldsymbol{D}(\omega, \boldsymbol{k}) \cdot \boldsymbol{E} = 0, \tag{6}$$

where $\boldsymbol{D}$ can be expressed in terms of the dielectric tensor $\boldsymbol{K}(\omega, \boldsymbol{k})$ as

$$\boldsymbol{D}(\omega, \boldsymbol{k}) = \boldsymbol{K}(\omega, \boldsymbol{k}) + (\boldsymbol{kk} - k^2\boldsymbol{I})\frac{c^2}{\omega^2}, \tag{7}$$

where $\boldsymbol{I}$ is the unit tensor and $c = 1/\sqrt{\epsilon_0\mu_0}$ is the speed of light. And the relation to conductivity tensor $\boldsymbol{\sigma}$ is

$$\boldsymbol{K} = \boldsymbol{I} - \boldsymbol{\sigma}/(i\omega\epsilon_0), \tag{8}$$

with the relation between current $\boldsymbol{J}$ and electric field $\boldsymbol{E}$ be

$$\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{9}$$

The dispersion relation is

$$|\boldsymbol{D}(\omega, \boldsymbol{k})| = \begin{vmatrix} K_{xx} - \frac{c^2k^2}{\omega^2}\cos^2\theta & K_{xy} & K_{xz} + \frac{c^2k^2}{\omega^2}\sin\theta\cos\theta \\ K_{yx} & K_{yy} - \frac{c^2k^2}{\omega^2} & K_{yz} \\ K_{zx} + \frac{c^2k^2}{\omega^2}\sin\theta\cos\theta & K_{zy} & K_{zz} - \frac{c^2k^2}{\omega^2}\sin^2\theta \end{vmatrix} = 0, \tag{10}$$

The standard linearized kinetic theory gives [Ichimaru1973, p51]

$$\boldsymbol{K}(\omega, \boldsymbol{k}) = \Big(1 - \frac{\omega_p^2}{\omega^2}\Big)\boldsymbol{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{n=-\infty}^{\infty} \int d\boldsymbol{v} \frac{\boldsymbol{\Pi}_s}{\omega - k_\parallel v_\parallel - n\Omega_s}\Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + k_\parallel \frac{\partial f_{s0}}{\partial v_\parallel}\Big), \tag{11}$$

and

$$\boldsymbol{\Pi}_s = \begin{pmatrix} (\frac{n\Omega_s}{k_\perp}J_n)^2 & i\frac{n\Omega_s}{k_\perp}v_\perp J_n J_n' & \frac{n\Omega_s}{k_\perp}v_\parallel J_n^2 \\ -i\frac{n\Omega_s}{k_\perp}v_\perp J_n J_n' & (v_\perp J_n')^2 & -iv_\perp v_\parallel J_n J_n' \\ \frac{n\Omega_s}{k_\perp}v_\parallel J_n^2 & iv_\perp v_\parallel J_n J_n' & (v_\parallel J_n)^2 \end{pmatrix}, \tag{12}$$

where $\int d\boldsymbol{v} \equiv 2\pi\int_0^\infty v_\perp dv_\perp \int_{-\infty}^\infty dv_\parallel$, Bessel function $J_n = J_n(\frac{k_\perp v_\perp}{\Omega_s})$ and $\omega_p^2 = \sum_s \omega_{ps}^2$.

For Bessel function, we have [Stix1992 p256]: $J_n'(x) = [J_{n-1}(x) - J_{n+1}(x)]/2$, $nJ_n(x)/x = [J_{n-1}(x) + J_{n+1}(x)]/2$, $\sum_{n=-\infty}^\infty J_n^2 = 1$, $\sum_{n=-\infty}^\infty J_n J_n' = 0$, $\sum_{n=-\infty}^\infty nJ_n^2 = 0$, $\sum_{n=-\infty}^\infty (J_n')^2 = \frac{1}{2}$, $\sum_{n=-\infty}^\infty \frac{n^2J_n^2(x)}{x^2} = \frac{1}{2}$, $\sum_{n=-\infty}^\infty nJ_nJ_n' = 0$, $J_n(-x) = J_{-n}(x) = (-1)^nJ_n(x)$, and $\sum_{m\neq0,n=-\infty}^\infty J_nJ_{n+m} = 0$.

1.3.2 Final dispersion relation

For the loss cone distribution function Eq.(1), we have

$$\boldsymbol{K}(\omega,\boldsymbol{k})=\Big(1-\frac{\omega_p^2}{\omega^2}\Big)\boldsymbol{I}+\sum_s\frac{\omega_{ps}^2}{\omega^2}\Big[\frac{1-\alpha_s\Delta_s}{1-\alpha_s}\boldsymbol{\chi}_{sa}+\frac{-\alpha_s+\alpha_s\Delta_s}{1-\alpha_s}\boldsymbol{\chi}_{sb}\Big], \tag{13}$$

with

$$\boldsymbol{\chi}_{s\sigma}=\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{\boldsymbol{\Pi}_s}{\omega-k_\parallel v_\parallel-n\Omega_s}\Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0\sigma}}{\partial v_\perp}+k_\parallel\frac{\partial f_{s0\sigma}}{\partial v_\parallel}\Big), \tag{14}$$

$\sigma=a,b$, and

$$\Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp}+k_\parallel\frac{\partial f_{s0}}{\partial v_\parallel}\Big)=-2\Big[\frac{n\Omega_s}{v_{\perp ts}^2}+\frac{k_\parallel(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0a}-2\Big[\frac{n\Omega_s}{\alpha_s v_{\perp ts}^2}+\frac{k_\parallel(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0b}, \tag{15}$$

i.e.,

$$\Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0\sigma}}{\partial v_\perp}+k_\parallel\frac{\partial f_{s0\sigma}}{\partial v_\parallel}\Big)=-2\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_\parallel(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma}. \tag{16}$$

Use

$$\int_{-\infty}^{\infty}\frac{f_z}{\omega-k_zv_\parallel-n\Omega_s}dv_\parallel=-\frac{1}{k_zv_{zts}}Z(\zeta_{sn}), \tag{17}$$

$$\int_{-\infty}^{\infty}\frac{k_z(v_\parallel-v_{ds})f_z}{\omega-k_zv_\parallel-n\Omega_s}dv_\parallel=-[1+\zeta_{sn}Z(\zeta_{sn})], \tag{18}$$

$$\int_{-\infty}^{\infty}\frac{[k_z(v_\parallel-v_{ds})]^2f_z}{\omega-k_zv_\parallel-n\Omega_s}dv_\parallel=-\omega_{sn}[1+\zeta_{sn}Z(\zeta_{sn})], \tag{19}$$

$$\int_{-\infty}^{\infty}\frac{[k_z(v_\parallel-v_{ds})]^3f_z}{\omega-k_zv_\parallel-n\Omega_s}dv_\parallel=-\frac{k_z^2v_{zts}^2}{2}-\omega_{sn}^2[1+\zeta_{sn}Z(\zeta_{sn})], \tag{20}$$

we have

$$\boldsymbol{\chi}_{s\sigma}=-2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{\boldsymbol{\Pi}_s}{\omega-k_zv_\parallel-n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_z(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma}, \tag{21}$$

and

$$
\begin{aligned}
\chi_{s\sigma}^{11} &= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{(\frac{n\Omega_s}{k_\perp}J_n)^2}{\omega-k_zv_\parallel-n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_z(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma} \qquad (22)\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n^3\Omega_s^3}{k_\perp^2v_{\perp ts\sigma}^2}J_n^2\frac{1}{k_zv_{zts}}Z(\zeta_{sn})+\frac{n^2\Omega_s^2}{k_\perp^2v_{zts\sigma}^2}J_n^2[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}f_{s0\perp\sigma}\\
&= 2\sum_{n=-\infty}^{\infty}\Big\{\frac{n^3\Omega_s^3}{k_\perp^2v_{\perp ts\sigma}^2}\frac{1}{k_zv_{zts}}Z(\zeta_{sn})+\frac{n^2\Omega_s^2}{k_\perp^2v_{zts\sigma}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\Gamma_n(b_{s\sigma})\\
&= \sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{k_zv_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n^2\Gamma_n(b_{s\sigma})}{b_{s\sigma}}\\
&= \sum_{n=-\infty}^{\infty}\Big\{1+\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n^2\Gamma_n(b_{s\sigma})}{b_{s\sigma}}\\
&= 1+\sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n^2\Gamma_n(b_{s\sigma})}{b_{s\sigma}},
\end{aligned}
$$

$$
\begin{aligned}
\chi_{s\sigma}^{12} &= -\chi_{s\sigma}^{21}=-2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{i\frac{n\Omega_s}{k_\perp}v_\perp J_nJ_n'}{\omega-k_zv_\parallel-n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_z(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma} \qquad (23)\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n\Omega_s}{k_zv_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}i\frac{n\Omega_s}{k_\perp v_{\perp ts\sigma}^2}v_\perp J_nJ_n'f_{s0\perp\sigma}\\
&= \sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{k_zv_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}ine^{-b_{s\sigma}}[I_n'(b_{s\sigma})-I_n(b_{s\sigma})]\\
&= \sum_{n=-\infty}^{\infty}\Big\{1+\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}ine^{-b_{s\sigma}}[I_n'-I_n]\\
&= \sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}in\Gamma_n',
\end{aligned}
$$

$$
\begin{aligned}
\chi_{s\sigma}^{22} &= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{(v_\perp J_n')^2}{\omega-k_zv_\parallel-n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_z(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma} \qquad (24)\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n\Omega_s}{k_zv_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{v_\perp^2}{v_{\perp ts\sigma}^2}(J_n')^2f_{s0\perp\sigma}\\
&= \sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{k_zv_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}e^{-b_{s\sigma}}[\frac{n^2}{b_{s\sigma}}I_n(b_{s\sigma})+2b_{s\sigma}I_n-2b_{s\sigma}I_n']\\
&= 1+\sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}[n^2\Gamma_n/b_{s\sigma}-2b_{s\sigma}\Gamma_n'],
\end{aligned}
$$

$$
\begin{aligned}
\chi^{13}_{s\sigma} &= -\chi^{31}_{s\sigma} = -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{\frac{n\Omega_s}{k_\perp}v_\parallel J_n^2}{\omega-k_zv_\parallel-n\Omega_s}\Big[\frac{n\Omega_s}{v^2_{\perp ts\sigma}}+\frac{k_z(v_\parallel-v_{ds})}{v^2_{zts}}\Big]f_{s0\sigma} \qquad (25)\\
&= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{\frac{n\Omega_s}{k_\perp}J_n^2\frac{1}{k_z}}{\omega-k_zv_\parallel-n\Omega_s}[k_z(v_\parallel-v_{ds})+k_zv_{ds}]\Big[\frac{n\Omega_s}{v^2_{\perp ts\sigma}}+\frac{k_z(v_\parallel-v_{ds})}{v^2_{zts}}\Big]f_{s0\sigma}\\
&= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{\frac{n\Omega_s}{k_\perp}J_n^2\frac{1}{k_z}}{\omega-k_zv_\parallel-n\Omega_s}\Big[k_zv_{ds}\frac{n\Omega_s}{v^2_{\perp ts\sigma}}+\Big(\frac{k_zv_{ds}}{v^2_{zts}}+\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\Big)k_z(v_\parallel-v_{ds})+\frac{k_z^2(v_\parallel-v_{ds})^2}{v^2_{zts}}\Big]f_{s0\sigma}\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn})+\Big(\frac{k_zv_{ds}}{v^2_{zts}}+\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]+\\
&\quad \frac{1}{v^2_{zts}}\omega_{sn}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n\Omega_s}{k_\perp}J_n^2\frac{1}{k_z}f_{s0\perp\sigma}\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn})+\Big(\frac{\omega-(1-\lambda_{Ts\sigma})n\Omega_s}{v^2_{zts}}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n\Omega_s}{k_\perp}J_n^2\frac{1}{k_z}f_{s0\perp\sigma}\\
&= 2\sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn})+\Big(\frac{\omega-(1-\lambda_{Ts\sigma})n\Omega_s}{v^2_{zts}}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{n\Omega_s}{k_\perp}\frac{1}{k_z}\Gamma_n\\
&= 2\sum_{n=-\infty}^{\infty}\Big[\frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\frac{\omega}{k_zv_{zts}}+\Big\{\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\eta_{sn}\Big]\frac{n\Omega_s}{k_\perp}\frac{1}{k_z}\Gamma_n\\
&= \sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn})+\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{\sqrt{2\lambda_{Ts\sigma}}}{a_{s\sigma}}\eta_{sn}n\Gamma_n,
\end{aligned}
$$

We hav used: $\Big\{\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn})+\Big(\frac{\omega-(1-\lambda_{Ts\sigma})n\Omega_s}{v^2_{zts}}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\} = \Big\{\frac{n\Omega_s}{v^2_{\perp ts\sigma}}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn})+\frac{k_z}{v_{\perp ts\sigma}\sqrt{\lambda_{Ts\sigma}}}\Bigg(\frac{\omega-n\Omega_s+\lambda_{Ts\sigma}n}{k_zv_{zts}}$ $\zeta_{sn}Z(\zeta_{sn})]\Big\} = \frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\Big\{n\Omega_s\frac{k_zv_{ds}}{k_z^2v^2_{zts}}Z(\zeta_{sn})+\frac{1}{\lambda_{Ts\sigma}}\Big(\eta_{sn}+\frac{\lambda_{Ts\sigma}n\Omega_s}{k_zv_{zts}}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\} = \frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\Big\{n\Omega_s\frac{k_zv_{ds}}{k_z^2v^2_{zts}}Z(\zeta_{sn})+$ $\eta_{sn}\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]+\eta_{sn}[1+\zeta_{sn}Z(\zeta_{sn})]+\frac{n\Omega_s}{k_zv_{zts}}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\} = \frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\Big\{\frac{\omega}{k_zv_{zts}}+$ $n\Omega_s\frac{k_zv_{ds}}{k_z^2v^2_{zts}}Z(\zeta_{sn})+\eta_{sn}\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]+\frac{\omega}{k_zv_{zts}}\zeta_{sn}Z(\zeta_{sn})\Big\} = \frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\frac{\omega}{k_zv_{zts}}+\frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\eta_{sn}\Big\{\zeta_{s0}Z(\zeta_{sn})+$ $\Big(\frac{1}{\lambda_{Ts\sigma}}-1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}$, and $\sum_{n=-\infty}^{\infty}n\Gamma_n=0$.

$$
\begin{aligned}
\chi^{32}_{s\sigma} &= -\chi^{32}_{s\sigma} = -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{iv_\perp v_\parallel J_n J_n'}{\omega - k_z v_\parallel - n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2} + \frac{k_z(v_\parallel - v_{ds})}{v_{zts}^2}\Big] f_{s0\sigma} \qquad (26)\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp \Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn}) + \Big(\frac{\omega-(1-\lambda_{Ts\sigma})n\Omega_s}{v_{zts}^2}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\} iv_\perp J_n J_n' \frac{1}{k_z} f_{s0\perp\sigma}\\
&= \sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{v_{ds}}{v_{zts}}Z(\zeta_{sn}) + \Big(\frac{\omega-(1-\lambda_{Ts\sigma})n\Omega_s}{v_{zts}^2}\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{ik_\perp v_{\perp ts\sigma}^2}{k_z\Omega_s} e^{-b_{s\sigma}}[I_n' - I_n]\\
&= \sum_{n=-\infty}^{\infty}\Big[\frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\frac{\omega}{k_z v_{zts}} + \Big\{\zeta_{s0}Z(\zeta_{sn}) + \Big(\frac{1}{\lambda_{Ts\sigma}} - 1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{k_z\sqrt{\lambda_{Ts\sigma}}}{v_{\perp ts\sigma}}\eta_{sn}\Big]\frac{ik_\perp v_{\perp ts\sigma}^2}{k_z\Omega_s}\Gamma_n'\\
&= \sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn}) + \Big(\frac{1}{\lambda_{Ts\sigma}} - 1\Big)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\sqrt{2\lambda_{Ts\sigma}}\, i\eta_{sn} a_{s\sigma}\Gamma_n',
\end{aligned}
$$

we have used $\sum_{n=-\infty}^{\infty}\Gamma_n' = 0$.

$$
\begin{aligned}
\chi^{33}_{s\sigma} &= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{(v_\parallel J_n)^2}{\omega - k_z v_\parallel - n\Omega_s}\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2} + \frac{k_z(v_\parallel - v_{ds})}{v_{zts}^2}\Big] f_{s0\sigma} \qquad (27)\\
&= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{J_n^2\frac{1}{k_z^2}}{\omega - k_z v_\parallel - n\Omega_s}[k_z^2(v_\parallel - v_{ds})^2 + 2k_z v_{ds}k_z(v_\parallel - v_{ds}) + k_z^2 v_{ds}^2]\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2} + \frac{k_z(v_\parallel - v_{ds})}{v_{zts}^2}\Big] f_{s0\sigma}\\
&= -2\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{J_n^2\frac{1}{k_z^2}}{\omega - k_z v_\parallel - n\Omega_s}\Big[k_z^2 v_{ds}^2\frac{n\Omega_s}{v_{\perp ts\sigma}^2} + \Big(\frac{k_z^2 v_{ds}^2}{v_{zts}^2} + \frac{2k_z v_{ds} n\Omega_s}{v_{\perp ts\sigma}^2}\Big)k_z(v_\parallel - v_{ds})\\
&\quad + \Big(\frac{2k_z v_{ds}}{v_{zts}^2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\Big)k_z^2(v_\parallel - v_{ds})^2 + \frac{k_z^3(v_\parallel - v_{ds})^3}{v_{zts}^2}\Big] f_{s0\sigma}\\
&= 4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp\Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \Big(\frac{k_z^2 v_{ds}^2}{v_{zts}^2} + \frac{2k_z v_{ds} n\Omega_s}{v_{\perp ts\sigma}^2}\Big)[1+\zeta_{sn}Z(\zeta_{sn})] +\\
&\quad \Big(\frac{2k_z v_{ds}}{v_{zts}^2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\Big)\omega_{sn}[1+\zeta_{sn}Z(\zeta_{sn})] + \frac{k_z^2}{2} + \frac{\omega_{sn}^2}{v_{zts}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\} J_n^2\frac{1}{k_z^2} f_{s0\perp\sigma}\\
&= 2\sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \Big(\frac{k_z^2 v_{ds}^2}{v_{zts}^2} + \frac{2k_z v_{ds} n\Omega_s}{v_{\perp ts\sigma}^2}\Big)[1+\zeta_{sn}Z(\zeta_{sn})] +\\
&\quad \Big(\frac{2k_z v_{ds}}{v_{zts}^2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\Big)\omega_{sn}[1+\zeta_{sn}Z(\zeta_{sn})] + \frac{k_z^2}{2} + \frac{\omega_{sn}^2}{v_{zts}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\frac{1}{k_z^2}\Gamma_n\\
&= 2\sum_{n=-\infty}^{\infty}\Big\{\frac{k_z^2}{2} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}]\\
&\quad + k_z^2\lambda_{Ts\sigma}\eta_{sn}^2\Big(\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\Big)\Big\}\frac{1}{k_z^2}\Gamma_n\\
&= 1 + 2\lambda_{Ts\sigma}\eta_{s0}^2 + \sum_{n=-\infty}^{\infty}\Big\{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}2\lambda_{Ts\sigma}\eta_{sn}^2\Gamma_n,
\end{aligned}
$$

We have used: $\left\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \left(\frac{k_z^2 v_{ds}^2}{v_{zts}^2} + \frac{2k_z v_{ds} n\Omega_s}{v_{\perp ts\sigma}^2}\right)[1+\zeta_{sn}Z(\zeta_{sn})] + \left(\frac{2k_z v_{ds}}{v_{zts}^2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\right)\omega_{sn}[1+\zeta_{sn}Z(\zeta_{sn})] + \frac{k_z^2}{2} + \frac{\omega_{sn}^2}{v_{zts}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \left[\left(\frac{k_z^2 v_{ds}^2}{v_{zts}^2} + \frac{2k_z v_{ds} n\Omega_s}{v_{\perp ts\sigma}^2}\right) + \left(\frac{2k_z v_{ds}}{v_{zts}^2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\right)\omega_{sn} + \frac{\omega_{sn}^2}{v_{zts}^2}\right][1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \frac{1}{v_{zts}^2}[(k_z^2 v_{ds}^2 + \lambda_{Ts\sigma}2k_z v_{ds} n\Omega_s) + (2k_z v_{ds} + \lambda_{Ts\sigma}n\Omega_s)\omega_{sn} + \omega_{sn}^2][1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \frac{1}{v_{zts}^2}[(\omega - n\Omega_s)^2 + \lambda_{Ts\sigma}n\Omega_s(\omega - n\Omega_s + k_z v_{ds})][1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[(\omega - n\Omega_s)^2 + n\Omega_s(\omega - n\Omega_s + k_z v_{ds})][1+\zeta_{sn}Z(\zeta_{sn})] + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega - n\Omega_s)^2(1/\lambda_{Ts\sigma} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}}Z(\zeta_{sn}) + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}][1+\zeta_{sn}Z(\zeta_{sn})] + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega - n\Omega_s)^2(1/\lambda_{Ts\sigma} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}] + \left[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{k_z v_{ds}^2}{v_{zts}} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds})\zeta_{sn}\right]Z(\zeta_{sn}) + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega - n\Omega_s)^2(1/\lambda_{Ts\sigma} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}] + \frac{\lambda_{Ts\sigma}}{k_z v_{zts}^3}\left[n\Omega_s k_z^2 v_{ds}^2 + (\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds})(\omega - k_z v_{ds} - n\Omega_s)\right]Z(\zeta_{sn}) + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega - n\Omega_s)^2(1/\lambda_{Ts\sigma} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}] + \frac{\lambda_{Ts\sigma}}{k_z v_{zts}^3}(\omega - n\Omega_s)^2(\omega - k_z v_{ds})Z(\zeta_{sn}) + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}(\omega - n\Omega_s)^2(1/\lambda_{Ts\sigma} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right\} = \left\{\frac{k_z^2}{2} + \frac{\lambda_{Ts\sigma}}{v_{zts}^2}[\omega^2 - n\Omega_s\omega + n\Omega_s k_z v_{ds}] + k_z^2\lambda_{Ts\sigma}\eta_{sn}^2\left(\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\right)\right\}$, $\sum_{n=-\infty}^{\infty} n\Gamma_n = 0$ and $\sum_{n=-\infty}^{\infty}\Gamma_n = 1$.

Thus, we have

$$\boldsymbol{K}(\omega,\boldsymbol{k}) = \boldsymbol{I} + \sum_s \frac{\omega_{ps}^2}{\omega^2}\sum_{\sigma=a,b} r_{s\sigma}\Big[\sum_{n=-\infty}^{\infty}\{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1+\zeta_{sn}Z(\zeta_{sn})]\}\boldsymbol{X}_{\sigma n} + 2\eta_{s0}^2\lambda_{Ts\sigma}\boldsymbol{L}\Big], \tag{28}$$

with

$$\boldsymbol{X}_{\sigma n} = \begin{pmatrix} n^2\Gamma_n/b_{s\sigma} & in\Gamma_n' & (2\lambda_{Ts\sigma})^{1/2}\eta_{sn}\frac{n}{a_{s\sigma}}\Gamma_n \\ -in\Gamma_n' & n^2\Gamma_n/b_{s\sigma} - 2b_{s\sigma}\Gamma_n' & -i(2\lambda_{Ts\sigma})^{1/2}\eta_{sn}a_{s\sigma}\Gamma_n' \\ (2\lambda_{Ts\sigma})^{1/2}\eta_{sn}\frac{n}{a_{s\sigma}}\Gamma_n & i(2\lambda_{Ts\sigma})^{1/2}\eta_{sn}a_{s\sigma}\Gamma_n' & 2\lambda_{Ts\sigma}\eta_{sn}^2\Gamma_n \end{pmatrix}, \tag{29}$$

$r_{sa} = \left(\frac{1-\alpha_s\Delta_s}{1-\alpha_s}\right)$, $r_{sb} = \left(\frac{-\alpha_s+\alpha_s\Delta_s}{1-\alpha_s}\right)$, and the matrix components of $\boldsymbol{L}$ are all zero except for $L_{zz} = 1$.

Note: If $\alpha_s = 1$, to avoid the singularity, we should set $r_{sa} = 1$ and $r_{sb} = 0$ in the code.

If no loss cone, i.e., $r_{sb} = 0$, the $\sum_{\sigma=a,b}$ is only $\sum_{\sigma=a}$ with $r_{sa} = 1$ and the above dispersion relation reduces to the drift bi-Maxwellian version [Miyamoto2004 p210] used in previous PDRK.

1.4 The Linear Transformation

Consider that the $\boldsymbol{K}(\omega,\boldsymbol{k})$ and $\boldsymbol{X}_{\sigma n}$ is very similar to the previous version drift bi-Maxwellian PDRK, the linear transformation could also be very similar, and we can follow the original PDRK derivation directly.

To seek an equivalent linear system, the Maxwells equations

$$\partial_t\boldsymbol{E} = c^2\nabla\times\boldsymbol{B} - \boldsymbol{J}/\epsilon_0, \tag{30a}$$

$$\partial_t\boldsymbol{B} = -\nabla\times\boldsymbol{E}, \tag{30b}$$

do not need to be changed. We only need to seek a new linear system for $\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}$. It is easy to find that after J-pole expansion, the relations between $\boldsymbol{J}$ and $\boldsymbol{E}$ has the following form

$$\left(\begin{array}{c} J_x \\ J_y \\ J_z \end{array} \right) = \left(\begin{array}{ccc} a_{11} + \sum_{snjm} \frac{b_{snjm11}}{\omega - c_{snjm11}} & a_{12} + \sum_{snjm} \frac{b_{snjm12}}{\omega - c_{snjm12}} & a_{13} + \sum_{snjm} \frac{b_{snjm13}}{\omega - c_{snjm13}} \\ a_{21} + \sum_{snjm} \frac{b_{snjm21}}{\omega - c_{snjm21}} & a_{22} + \sum_{snjm} \frac{b_{snjm22}}{\omega - c_{snjm22}} & a_{23} + \sum_{snjm} \frac{b_{snjm23}}{\omega - c_{snjm23}} \\ a_{31} + \sum_{snjm} \frac{b_{snjm31}}{\omega - c_{snjm31}} & a_{32} + \sum_{snjm} \frac{b_{snjm32}}{\omega - c_{snjm32}} & a_{33} + \sum_{snjm} \frac{b_{snjm33}}{\omega - c_{snjm33}} + d_{33}\omega \end{array} \right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array} \right). \tag{31}$$

Fortunately, noting the relations in Z function ($\sum_j b_j = -1$, $\sum_j b_j c_j = 0$ and $\sum_j b_j c_j^2 = -1/2$) and in Bessel functions [$\sum_{n=-\infty}^{\infty} I_n(b) = e^b$, $\sum_{n=-\infty}^{\infty} n I_n(b) = 0$, $\sum_{n=-\infty}^{\infty} n^2 I_n(b) = b e^b$], we find that $a_{ij} = 0$ $(i, j = 1, 2, 3)$ and $d_{33} = 0$. Eq.(31) can be changed further to

$$\left(\begin{array}{c} J_x \\ J_y \\ J_z \end{array} \right) = -i\epsilon_0 \left(\begin{array}{ccc} \frac{b_{11}}{\omega} + \sum_{snj} \frac{b_{snj11}}{\omega - c_{snj}} & \frac{b_{12}}{\omega} + \sum_{snj} \frac{b_{snj12}}{\omega - c_{snj}} & \frac{b_{13}}{\omega} + \sum_{snj} \frac{b_{snj13}}{\omega - c_{snj}} \\ \frac{b_{21}}{\omega} + \sum_{snj} \frac{b_{snj21}}{\omega - c_{snj}} & \frac{b_{22}}{\omega} + \sum_{snj} \frac{b_{snj22}}{\omega - c_{snj}} & \frac{b_{23}}{\omega} + \sum_{snj} \frac{b_{snj23}}{\omega - c_{snj}} \\ \frac{b_{31}}{\omega} + \sum_{snj} \frac{b_{snj31}}{\omega - c_{snj}} & \frac{b_{32}}{\omega} + \sum_{snj} \frac{b_{snj32}}{\omega - c_{snj}} & \frac{b_{33}}{\omega} + \sum_{snj} \frac{b_{snj33}}{\omega - c_{snj}} \end{array} \right) \left(\begin{array}{c} E_x \\ E_y \\ E_z \end{array} \right). \tag{32}$$

Combining Eqs. (30) and (32), the equivalent linear system for (10) can be obtained as

$$\left\{ \begin{array}{rcl} \omega v_{snjx} &=& c_{snj} v_{snjx} + b_{snj11} E_x + b_{snj12} E_y + b_{snj13} E_z, \\ \omega j_x &=& b_{11} E_x + b_{12} E_y + b_{13} E_z, \\ i J_x \epsilon_0 &=& j_x + \sum_{snj} v_{snjx}, \\ \omega v_{snjy} &=& c_{snj} v_{snjy} + b_{snj21} E_x + b_{snj22} E_y + b_{snj23} E_z, \\ \omega j_y &=& b_{21} E_x + b_{22} E_y + b_{23} E_z, \\ i J_y/\epsilon_0 &=& j_y + \sum_{snj} v_{snjy}, \\ \omega v_{snjz} &=& c_{snj} v_{snjz} + b_{snj31} E_x + b_{snj32} E_y + b_{snj33} E_z, \\ \omega j_z &=& b_{31} E_x + b_{32} E_y + b_{33} E_z, \\ i J_z/\epsilon_0 &=& j_z + \sum_{snj} v_{snjz}, \\ \omega E_x &=& c^2 k_z B_y - i J_x/\epsilon_0, \\ \omega E_y &=& -c^2 k_z B_x + c^2 k_x B_z - i J_y/\epsilon_0, \\ \omega E_z &=& -c^2 k_x B_y - i J_z/\epsilon_0, \\ \omega B_x &=& -k_z E_y, \\ \omega B_y &=& k_z E_x - k_x E_z, \\ \omega B_z &=& k_x E_y, \end{array} \right. \tag{33}$$

which yields a sparse matrix eigenvalue problem. Again, the symbols v_{snjx}, $j_{x,y,z}$ and $J_{x,y,z}$ used here do not have direct physical meanings but are analogy to the perturbed velocity and current density in the fluid derivations of plasma waves. The elements of the eigenvector $(E_x, E_y, E_z, B_x, B_y, B_z)$ still represent the original electric and magnetic fields. Thus, the polarization of the solutions can also be obtained in a straightforward manner. The dimension of the matrix is $NN = 3 \times (SNJ +$

$1) + 6 = 3 \times [S \times (2 \times N + 1) \times J + 1] + 6$. The coefficients are

$$\left\{\begin{array}{ccc} b_{snj11} & = & \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j(1-k_z b_{j0}/c_{snj})n^2\Gamma_n/b_s, \\ b_{11} & = & \sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j(k_z b_{j0}/c_{snj})n^2\Gamma_n/b_s, \\ b_{snj12} & = & \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j(1-k_z b_{j0}/c_{snj})in\Gamma_n', \\ b_{12} & = & \sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j(k_z b_{j0}/c_{snj})in\Gamma_n', \\ b_{snj21}=-b_{snj12} & , & b_{21}=-b_{12}, \\ b_{snj22} & = & \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j(1-k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s-2b_s\Gamma_n'), \\ b_{22} & = & \sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j(k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s-2b_s\Gamma_n'), \\ b_{snj13} & = & \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j[c_j/\lambda_{Ts}+n\omega_{cs}b_{j0}/(c_{snj}v_{zts})]n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\ b_{13} & = & -\sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j[n\omega_{cs}b_{j0}/(c_{snj}v_{zts})]n\sqrt{2\lambda_{Ts}}\Gamma_n/\alpha_s, \\ b_{snj31}=b_{snj13} & , & b_{31}=b_{13}, \\ b_{snj23} & = & -i\sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j[c_j/\lambda_{Ts}+n\omega_{cs}b_{j0}/(c_{snj}v_{zts})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\ b_{23} & = & i\sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j[n\omega_{cs}b_{j0}/(c_{snj}v_{zts})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s, \\ b_{snj32}=-b_{snj23} & , & b_{32}=-b_{23}, \\ b_{snj33} & = & \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j[(c_j/\lambda_{Ts}+b_{j0}n\omega_{cs}/(c_{snj}v_{zts})](v_{ds}/v_{zts}+c_j)2\lambda_{Ts}\Gamma_n, \\ b_{33} & = & \sum_\sigma r_{s\sigma}\sum_{snj}\omega_{ps}^2 b_j[n^2\omega_{cs}^2 b_{j0}/(c_{snj}v_{zts}^2 k_z)]2\lambda_{Ts}\Gamma_n, \\ c_{snj} & = & k_z c_j v_{zts}+k_z v_{ds}+n\omega_{cs}, \end{array}\right. \tag{34}$$

where $b_{j0\sigma} = v_{ds} + (1 - 1/\lambda_{Ts\sigma})c_j v_{zts}$. Note that we have used $b_{snj33} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j[(c_j/\lambda_{Ts\sigma} + b_{j0}n\omega_{cs}/(c_{snj}v_{zts})](v_{ds}/v_{zts} + c_j)2\lambda_{Ts}\Gamma_n$, instead of the original $b_{snj33} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j[(v_{ds}/v_{zts} + c_j)c_j/\lambda_{Ts} + n\omega_{cs}b_{j0}((1 - n\omega_{cs}/c_{snj})/v_{zts}^2)/k_z]2\lambda_{Ts}\Gamma_n$, though they are equivalent (see appendix).

Note the only difference to the previous drift bi-Maxwellian PDRK is the $\sum_\sigma r_{s\sigma}$ for $\sigma = a, b$. If no loss cone, i.e., $r_{sb} = 0$, the $\sum_{\sigma=a,b}$ is only $\sum_{\sigma=a}$ with $r_{sa} = 1$ and the above linear transformation matrix reduces to the previous drift bi-Maxwellian version PDRK [Xie2016].

If $a_{ij} \neq 0$, then the equivalent linear transformation is still straightforward. If $d_{33} \neq 0$, then the equivalent linear transformation will be more complicated. For our purposes, we do not need to discuss these cases.

1.5 The polarizations

2018-10-17 14:30

PDRK can obtain $(E_x, E_y, E_z, B_x, B_y, B_z)$ directly[1] from the matrix eigenvalue problem. Considered that the magnitude of the wave has no meaning for a linear system, we should do normalizations. We set $|E| = 1mV/m$ and $E_x = Re(E_x)$.

Some other useful: electric field energy $U_E = \frac{\epsilon_0}{2}\boldsymbol{E}\cdot\boldsymbol{E}^*$, magnetic field energy $U_B = \frac{1}{2\mu_0}\boldsymbol{B}\cdot\boldsymbol{B}^*$, energy flux Poynting vector $\boldsymbol{S} = \frac{1}{\mu_0}\boldsymbol{E}\times\boldsymbol{B}^*$. $P = E_y/iE_x$.

2 Benchmark

Based on above derivations with loss cone, we have updated PDRK to a new version pdrk-v2. A benchmark of this new version with PDRK is shown in Fig.1, with $\beta_{//p} = \beta_{//e} = 1$, $T_{p,\perp}/T_{p,//} = 2$

[1]In principle, PDRK can also obtain (E_x, E_y, E_z) as in standard 3×3 matrix $\boldsymbol{D}\cdot\boldsymbol{E} = 0$. To obtain group velocity $\boldsymbol{v}_g = d\omega/d\boldsymbol{k}$ or do ray tracing, we may also need $\partial D/\partial\omega$ and $\partial D/\partial\boldsymbol{k}$.

and $T_{e,\perp}/T_{e,//} = 1$. Good agreement is obtained. We should noticed that PDRK can give all the important solutions at once, whereas WHAMP requires carefully setting initial guess for root finding.

The input data in pdrk.in are

qs(e)	ms(mp)	ns(m^-3)	Tzs(eV)	Tps(eV)	alphas	Deltas	vds/c
1	1	1.e6	24840.	49680.	0.5	0.1	0.0
-1	5.447e-4	1.e6	24840.	24840.	0.5	0.1	0.0

and magnetic field B0=100.0E-9 T.

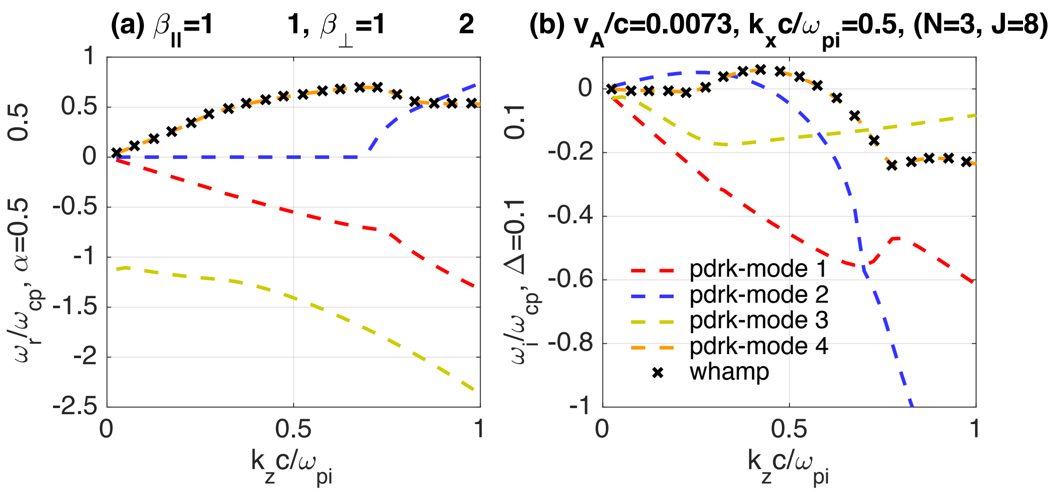

Figure 1: The pdrk-v2 agrees with WHAMP for the mirror mode with loss cone at $k_\perp c/\omega_{pi} = 0.5$, with loss cone parameters $\alpha_{e,i} = 0.5$ and $\Delta_{e,i} = 0.1$.

A Useful integrals

Plasma dispersion function $Z(\zeta)$

$$Z(\zeta) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{1}{x-\zeta} e^{-x^2} dx,$$

$$\frac{dZ}{d\zeta} = -2(1+\zeta Z),$$

$$\frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{x^2}{x-\zeta} e^{-x^2} dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{(x-\zeta)(x+\zeta)+\zeta^2}{x-\zeta} e^{-x^2} dx = \zeta + \zeta^2 Z(\zeta),$$

and some other useful integrals

$$\int_0^\infty x^{2n}e^{-x^2/a^2}dx = \frac{a^{2n+1}(2n-1)!!}{2^{n+1}}\sqrt{\pi},$$
$$\int_0^\infty x^{2n+1}e^{-x^2/a^2}dx = \frac{n!}{2}a^{2n+2},$$
$$\int_0^\infty xe^{-x^2/2}J_p^2(\sqrt{b}x)dx = e^{-b}I_p(b) = \Gamma_p(b),$$
$$\int_0^\infty x^3e^{-x^2/2}J_p^2(\sqrt{b}x)dx = 2e^{-b}[(1-b+p)I_p(b)+bI_{p+1}(b)],$$

where J_p and I_p are Bessel and modified Bessel functions, respectively. Note the integral range is from 0 to ∞. We can further obtain

$$\int_0^\infty x^2e^{-x^2/2}J_pJ_p'dx \equiv \int_0^\infty x^2e^{-x^2/2}J_p(\sqrt{b}x)\frac{dJ_p(\sqrt{b}x)}{d(\sqrt{b}x)}dx = \int_0^\infty x^2e^{-x^2/2}\frac{1}{2\sqrt{b}}\frac{d(J_p^2)}{dx}dx$$
$$= x^2e^{-x^2/2}\frac{1}{2\sqrt{b}}J_p^2\Big|_0^\infty - \int_0^\infty \frac{1}{2\sqrt{b}}\frac{d(x^2e^{-x^2/2})}{dx}J_p^2dx = \frac{1}{2\sqrt{b}}\int_0^\infty (x^3-2x)e^{-x^2/2}J_p^2dx$$
$$= \frac{1}{\sqrt{b}}\Big\{e^{-b}[(1-b+p)I_p(b)+bI_{p+1}(b)] - e^{-b}I_p(b)\Big\} = \frac{e^{-b}}{\sqrt{b}}[(-b+p)I_p(b)+bI_{p+1}(b)] = \cdots.$$

Note: $\Gamma_n'(b) = (I_n' - I_n)e^{-b}$, $I_n'(b) = (I_{n+1}+I_{n-1})/2$, $I_{-n} = I_n$. Fortunately, noting the relations in Z function ($\sum_j b_j = -1$, $\sum_j b_jc_j = 0$ and $\sum_j b_jc_j^2 = -1/2$) and in Bessel functions [$\sum_{n=-\infty}^{\infty} I_n(b) = e^b$, $\sum_{n=-\infty}^{\infty} nI_n(b) = 0$, $\sum_{n=-\infty}^{\infty} n^2I_n(b) = be^b$].

Note: $\sum_{n=-\infty}^{\infty} n\Gamma_n' = e^{-b}\sum_{n=-\infty}^{\infty} n(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = e^{-b}\sum_{n=1}^{\infty} n(\frac{I_{n+1}+I_{n-1}-I_{-n+1}-I_{-n-1}}{2} - I_n + I_{-n}) = 0$; $\sum_{n=-\infty}^{\infty} n\Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} nI_n = e^{-b}\sum_{n=1}^{\infty} n(I_n - I_{-n}) = 0$; $\sum_{n=-\infty}^{\infty}\Gamma_n' = e^{-b}\sum_{n=-\infty}^{\infty}(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = 0$; $\sum_{n=-\infty}^{\infty}\Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} I_n = 1$;

From [Stix1992 p257], we have

$$\frac{1}{\pi w^2}\int_0^\infty 2\pi v dv J_n^2\Big(\frac{kv}{\Omega}\Big)e^{-v^2/w^2} = e^{-\lambda}I_n(\lambda),$$
$$\frac{1}{\pi w^2}\int_0^\infty 2\pi v^2 dv J_n\Big(\frac{kv}{\Omega}\Big)J_n'\Big(\frac{kv}{\Omega}\Big)e^{-v^2/w^2} = -\frac{kw^2}{2\Omega}e^{-\lambda}[I_n(\lambda) - I_n'(\lambda)],$$
$$\frac{1}{\pi w^2}\int_0^\infty 2\pi v^3 dv \Big[J_n'\Big(\frac{kv}{\Omega}\Big)\Big]^2 e^{-v^2/w^2} = \frac{w^2}{2}e^{-\lambda}\Big[\frac{n^2}{\lambda}I_n(\lambda) + 2\lambda I_n - 2\lambda I_n'\Big],$$

with $\lambda = \frac{k^2w^2}{2\Omega^2}$.

B The electromagnetic dispersion relation

B.1 Basic idea

Firstly [Gurnett2005 sec.9.3]

$$\boldsymbol{k}\times(\boldsymbol{k}\times\boldsymbol{E}) + \frac{\omega^2}{c^2}\boldsymbol{K}\cdot\boldsymbol{E} = 0, \tag{35}$$

where $\boldsymbol{E}$ is the electric field of the wave and $\boldsymbol{K}$ is the dielectric tensor. And the current density

$$\boldsymbol{J} = \boldsymbol{\sigma} \cdot \boldsymbol{E}. \tag{36}$$

Once the conductivity tensor $\boldsymbol{\sigma}$ is known, the dielectric tensor can be computed using $\boldsymbol{K} = \boldsymbol{I} - \boldsymbol{\sigma}/(i\omega\epsilon_0)$. Compared with Eq.(10), we find that for EM3D dispersion relation

$$\boldsymbol{\sigma} = -i\epsilon_0 \sum_s \frac{\omega_{ps}^2}{\omega} \sum_{\sigma=a,b} r_{s\sigma} \Big[\sum_{n=-\infty}^{\infty} \{\zeta_{s0} Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1 + \zeta_{sn} Z(\zeta_{sn})]\} \boldsymbol{X}_{\sigma n} + 2\eta_{s0}^2 \lambda_{Ts\sigma} \boldsymbol{L} \Big]. \tag{37}$$

Combine the above equation and

$$Z(\zeta) \simeq Z_J(\zeta) = \sum_{j=1}^{J} \frac{b_j}{\zeta - c_j}, \tag{38}$$

we can obtain Eq.(31) and the final transform matrix. Here, b_j and c_j are constants for given J, as given in [Xie2016].

B.2 More details

Note

$$\frac{1}{\omega} \frac{b}{\omega - c} = \frac{b}{c} \Big(\frac{1}{\omega - c} - \frac{1}{\omega} \Big),$$

we have

$$\begin{aligned}
Y &\equiv \zeta_0 Z(\zeta_n) - (1 - \frac{1}{\lambda_T})[1 + \zeta_n Z(\zeta_n)] \\
&= \sum_{j=1}^{J} \frac{\zeta_0 b_j}{\zeta_n - c_j} - (1 - \frac{1}{\lambda_T}) \Big[1 + \sum_{j=1}^{J} \frac{\zeta_n b_j}{\zeta_n - c_j} \Big] \\
&= \sum_{j=1}^{J} b_j \Big[1 + \frac{c_j k_z v_{zts} + n\Omega_s}{\omega - c_{snj}} \Big] - (1 - \frac{1}{\lambda_T}) \Big[1 + \sum b_j + \sum_{j=1}^{J} \frac{b_j c_j}{\zeta_n - c_j} \Big] \\
&= -1 + \sum_{j=1}^{J} \frac{b_j c_j k_z v_{zts} + b_j n\Omega_s}{\omega - c_{snj}} - (1 - \frac{1}{\lambda_T}) \sum_{j=1}^{J} \frac{b_j c_j k_z v_{zts}}{\omega - c_{snj}} \\
&= -1 + \sum_{j=1}^{J} \frac{b_j (c_j k_z v_{zts}/\lambda_T + n\Omega_s)}{\omega - c_{snj}},
\end{aligned}$$

where we have defined $c_{snj} = k_z v_{ds} + n\Omega_s + k_z v_{zts} c_j$ and used $\sum_{j=1}^J b_j = -1$, and thus

$$\begin{aligned}
A &\equiv \frac{Y}{\omega} = -\frac{1}{\omega} + \frac{1}{\omega}\sum_{j=1}^J \frac{b_j(c_j k_z v_{zts}/\lambda_T + n\Omega_s)}{\omega - c_{snj}} \\
&= -\frac{1}{\omega} + \sum_{j=1}^J \frac{b_j(c_j k_z v_{zts}/\lambda_T + n\Omega_s)/c_{snj}}{\omega - c_{snj}} - \sum_{j=1}^J \frac{b_j(c_j k_z v_{zts}/\lambda_T + n\Omega_s)/c_{snj}}{\omega} \\
&= \sum_{j=1}^J \frac{b_j(c_j k_z v_{zts}/\lambda_T + n\Omega_s)/c_{snj}}{\omega - c_{snj}} + \sum_{j=1}^J \frac{b_j[1-(c_j k_z v_{zts}/\lambda_T + n\Omega_s)/c_{snj}]}{\omega} \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big[\frac{(c_j k_z v_{zts}/\lambda_T + n\Omega_s)}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big] = \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big(\frac{c_{snj} - k_z b_{j0}}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big),
\end{aligned}$$

where we have defined $b_{j0} = v_{ds} + (1 - 1/\lambda_T)c_j v_{zts}$, and note that $c_j k_z v_{zts}/\lambda_T + n\Omega_s = c_{snj} - k_z b_{j0}$. And

$$\begin{aligned}
\eta_n A &= \frac{\omega - n\Omega_s}{k_z v_{zts}} \sum_{j=1}^J \frac{b_j}{c_{snj}}\Big(\frac{c_{snj} - k_z b_{j0}}{\omega - c_{snj}} + \frac{k_z b_{j0}}{\omega}\Big) \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj} k_z v_{zts}}\Big[(c_{snj} - k_z b_{j0})\Big(1 + \frac{k_z v_{ds} + k_z v_{zts} c_j}{\omega - c_{snj}}\Big) + k_z b_{j0} - \frac{k_z b_{j0} n\Omega_s}{\omega}\Big] \\
&= \sum_{j=1}^J \frac{b_j}{c_{snj} k_z v_{zts}}\Big[(c_{snj} - k_z b_{j0})\frac{(k_z v_{ds} + k_z v_{zts} c_j)}{\omega - c_{snj}} - \frac{k_z b_{j0} n\Omega_s}{\omega} + (c_{snj} - k_z b_{j0}) + k_z b_{j0}\Big] \\
&= -\frac{1}{k_z v_{zts}} + \sum_{j=1}^J \frac{b_j}{c_{snj} v_{zts}}\Big[(c_{snj} - k_z b_{j0})\frac{(v_{ds} + v_{zts} c_j)}{\omega - c_{snj}} - \frac{b_{j0} n\Omega_s}{\omega}\Big] \\
&= -\frac{1}{k_z v_{zts}} + \sum_{j=1}^J \frac{b_j}{c_{snj} v_{zts}}\Big(\frac{c_{snj} c_j v_{zts}/\lambda_T + n\Omega_s b_{j0}}{\omega - c_{snj}} - \frac{b_{j0} n\Omega_s}{\omega}\Big),
\end{aligned}$$

where we have used that $c_{snj}c_jv_{zts}/\lambda_T + n\Omega_s b_{j0} = (k_zv_{ds} + n\Omega_s + k_zv_{zts}c_j)c_jv_{zts}/\lambda_T + n\Omega_s v_{ds} + n\Omega_s(1-1/\lambda_T)c_jv_{zts} = (c_jk_zv_{zts}/\lambda_T + n\Omega_s)(v_{ds}+v_{zts}c_j) = (c_{snj} - k_zb_{j0})(v_{ds}+v_{zts}c_j)$. And

$$
\begin{aligned}
\eta_n^2 A &= \frac{\omega - n\Omega_s}{k_zv_{zts}}\Big\{-\frac{1}{k_zv_{zts}} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}v_{zts}}\Big[(c_{snj}-k_zb_{j0})\frac{(v_{ds}+v_{zts}c_j)}{\omega - c_{snj}} - \frac{b_{j0}n\Omega_s}{\omega}\Big]\Big\} \\
&= -\frac{\omega}{k_z^2v_{zts}^2} + \frac{n\Omega_s}{k_z^2v_{zts}^2} + (\omega - n\Omega_s)\sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{zts}^2}\Big[(c_{snj}-k_zb_{j0})\frac{(v_{ds}+v_{zts}c_j)}{\omega - c_{snj}} - \frac{b_{j0}n\Omega_s}{\omega}\Big] \\
&= -\frac{\omega}{k_z^2v_{zts}^2} + \frac{n\Omega_s}{k_z^2v_{zts}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{zts}^2}\Big[(c_{snj}-k_zb_{j0})(v_{ds}+v_{zts}c_j)\Big(1 + \frac{k_zv_{ds}+k_zv_{zts}c_j}{\omega - c_{snj}}\Big) \\
&\quad + \frac{b_{j0}n^2\Omega_s^2}{\omega} - b_{j0}n\Omega_s\Big] \\
&= -\frac{\omega}{k_z^2v_{zts}^2} + \frac{n\Omega_s}{k_z^2v_{zts}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{zts}^2}\Big[(c_{snj}-k_zb_{j0})\frac{k_z(v_{ds}+v_{zts}c_j)^2}{\omega - c_{snj}} + \frac{b_{j0}n^2\Omega_s^2}{\omega}\Big] + \sum_{j=1}^{J}\frac{b_jc_j}{k_zv_{zts}} \\
&= -\frac{\omega}{k_z^2v_{zts}^2} + \frac{n\Omega_s}{k_z^2v_{zts}^2} + \sum_{j=1}^{J}\frac{b_j}{c_{snj}k_zv_{zts}^2}\Big[(c_{snj}-k_zb_{j0})\frac{k_z(v_{ds}+v_{zts}c_j)^2}{\omega - c_{snj}} + \frac{b_{j0}n^2\Omega_s^2}{\omega}\Big],
\end{aligned}
$$

where we have noticed $(c_{snj}-k_zb_{j0})(v_{ds}+v_{zts}c_j) - b_{j0}n\Omega_s = (c_jk_zv_{zts}/\lambda_T + n\Omega_s)(v_{ds}+v_{zts}c_j) - b_{j0}n\Omega_s = (v_{ds}c_jk_zv_{zts}/\lambda_T + v_{ds}n\Omega_s + v_{zts}c_jc_jk_zv_{zts}/\lambda_T + v_{zts}c_jn\Omega_s) - n\Omega_sv_{ds} - n\Omega_s(1-1/\lambda_T)c_jv_{zts} = (v_{ds}c_jk_zv_{zts}/\lambda_T + v_{zts}c_jc_jk_zv_{zts}/\lambda_T + n\Omega_sc_jv_{zts}/\lambda_T) = c_{snj}c_jv_{zts}/\lambda_T$, and used $\sum_{j=1}^{J} b_jc_j = 0$.

Use Eq.(29) and compare with Eq.(31):

- A has no constant term, $\Rightarrow a_{11} = a_{12} = a_{21} = a_{22} = 0$.
- $\sum_{n=-\infty}^{\infty} n\Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} nI_n = e^{-b}\sum_{n=1}^{\infty} n(I_n - I_{-n}) = 0$, and η_nA has only constant term $-\frac{1}{k_zv_{zts}}$, $\Rightarrow a_{13} = a_{31} = 0$.
- $\sum_{n=-\infty}^{\infty} \Gamma_n' = e^{-b}\sum_{n=-\infty}^{\infty}(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = 0$, and η_nA has only constant term $-\frac{1}{k_zv_{zts}}$, $\Rightarrow$ $a_{23} = a_{32} = 0$.
- $\sum_{n=-\infty}^{\infty} \Gamma_n = e^{-b}\sum_{n=-\infty}^{\infty} I_n = 1$. Thus using η_n^2A, the first two terms of σ_{33}, i.e., terms ω^1 and ω^0 are, $-i\epsilon_0\sum_s\omega_{ps}^2\Big[\sum_n 2\lambda_T\Gamma_n\Big(-\frac{\omega}{k_z^2v_{zts}^2} + \frac{n\Omega_s}{k_z^2v_{zts}^2}\Big) + \frac{2\eta_0^2\lambda_T}{\omega}\Big] = -i\epsilon_0\sum_s\omega_{ps}^2\Big[-2\lambda_T\frac{\omega}{k_z^2v_{zts}^2} + \frac{2\omega\lambda_T}{k_z^2v_{zts}^2}\Big] = 0$, $\Rightarrow a_{33} = 0$ and $d_{33} = 0$.

Not used yet: $\sum_{n=-\infty}^{\infty} n\Gamma_n' = e^{-b}\sum_{n=-\infty}^{\infty} n(\frac{I_{n+1}+I_{n-1}}{2} - I_n) = e^{-b}\sum_{n=1}^{\infty} n(\frac{I_{n+1}+I_{n-1}-I_{-n+1}-I_{-n-1}}{2} - I_n + I_{-n}) = 0$.

After the above steps, we can obtain Eq.(32) and corresponding coefficients in Eq.(34). For examples

- $b_{11} = \sum_s\omega_{ps}^2\sum_{\sigma=a,b} r_{s\sigma}\sum_n\frac{n^2\Gamma_n}{b_{s\sigma}}\sum_{j=1}^{J}\frac{b_jk_zb_{j0\sigma}}{c_{snj}} = \sum_{snj}\sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2 b_j(k_zb_{j0\sigma}/c_{snj})n^2\Gamma_n/b_{s\sigma}$.
- $b_{snj11} = \sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2\frac{n^2\Gamma_n}{b_s}\frac{b_j(c_{snj}-k_zb_{j0})}{c_{snj}} = \sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2 b_j(1 - k_zb_{j0}/c_{snj})n^2\Gamma_n/b_s$.

- $b_{12} = \sum_s \omega_{ps}^2 \sum_{\sigma=a,b} r_{s\sigma} \sum_n in\Gamma_n' \sum_{j=1}^J \frac{b_j k_z b_{j0}}{c_{snj}} = \sum_{snj} \sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2 b_j (k_z b_{j0}/c_{snj}) in\Gamma_n'$.

- $b_{snj12} = \sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2 in\Gamma_n' \frac{b_j(c_{snj}-k_z b_{j0})}{c_{snj}} = \sum_{\sigma=a,b} r_{s\sigma}\omega_{ps}^2 b_j(1-k_z b_{j0}/c_{snj}) in\Gamma_n'$.

- $b_{snj21} = -b_{snj12}$, $b_{21} = -b_{12}$.

- $b_{22} = \sum_s \omega_{ps}^2 \sum_\sigma r_{s\sigma} \sum_n (n^2\Gamma_n/b_s - 2b_s\Gamma_n') \sum_{j=1}^J \frac{b_j k_z b_{j0}}{c_{snj}} = \sum_{snj}\sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j (k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n')$.

- $b_{snj22} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 (n^2\Gamma_n/b_s - 2b_s\Gamma_n') \frac{b_j(c_{snj}-k_z b_{j0})}{c_{snj}} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j (1-k_z b_{j0}/c_{snj})(n^2\Gamma_n/b_s - 2b_s\Gamma_n')$.

- $b_{13} = -\sum_s \omega_{ps}^2 \sum_\sigma r_{s\sigma} \sum_n n\Gamma_n(\sqrt{2\lambda_{Ts}}/\alpha_s) \sum_{j=1}^J \frac{b_j b_{j0} n\Omega_s}{c_{snj} v_{zts}}$
  $= -\sum_{snj}\sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [b_{j0} n\Omega_s/(c_{snj} v_{zts})]\sqrt{2\lambda_{Ts}} n\Gamma_n/\alpha_s$.

- $b_{snj13} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 n\Gamma_n(\sqrt{2\lambda_{Ts}}/\alpha_s)\frac{b_j}{c_{snj} v_{zts}}(c_{snj} c_j v_{zts}/\lambda_{Ts} + n\Omega_s b_{j0})$
  $= \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [c_j/\lambda_{Ts} + n\Omega_s b_{j0}/(c_{snj} v_{zts})]\sqrt{2\lambda_{Ts}} n\Gamma_n/\alpha_s$.

- $b_{snj31} = b_{snj13}$, $b_{31} = b_{13}$.

- $b_{23} = i\sum_s \omega_{ps}^2 \sum_\sigma r_{s\sigma} \sum_n \Gamma_n'(\sqrt{2\lambda_{Ts}}\alpha_s) \sum_{j=1}^J \frac{b_j b_{j0} n\Omega_s}{c_{snj} v_{zts}}$
  $= i\sum_{snj}\sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [b_{j0} n\Omega_s/(c_{snj} v_{zts})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s$.

- $b_{snj23} = -i\omega_{ps}^2 \sum_\sigma r_{s\sigma}\Gamma_n'(\sqrt{2\lambda_{Ts}}\alpha_s)\frac{b_j}{c_{snj} v_{zts}}(c_{snj} c_j v_{zts}/\lambda_{Ts} + n\Omega_s b_{j0})$
  $= -i\omega_{ps}^2 \sum_\sigma r_{s\sigma} b_j [c_j/\lambda_{Ts} + n\Omega_s b_{j0}/(c_{snj} v_{zts})]\sqrt{2\lambda_{Ts}}\Gamma_n'\alpha_s$.

- $b_{snj32} = -b_{snj23}$, $b_{32} = -b_{23}$.

- $b_{33} = \sum_s \omega_{ps}^2 \sum_\sigma r_{s\sigma} \sum_n 2\lambda_{Ts}\Gamma_n \sum_{j=1}^J \frac{b_j b_{j0} n^2\Omega_s^2}{c_{snj} k_z v_{zts}^2} = \sum_{snj}\sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [b_{j0} n^2\Omega_s^2/(c_{snj} k_z v_{zts}^2)] 2\lambda_{Ts}\Gamma_n$.

- $b_{snj33} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 2\lambda_{Ts}\Gamma_n \frac{b_j(c_{snj}-k_z b_{j0})(v_{ds}+v_{zts}c_j)^2}{c_{snj} v_{zts}^2} = \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j \frac{(c_{snj} c_j v_{zts}/\lambda_T + b_{j0} n\Omega_s)(v_{ds}+v_{zts}c_j)}{c_{snj} v_{zts}^2} 2\lambda_{Ts}\Gamma_n$
  $= \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [(c_j/\lambda_{Ts} + b_{j0} n\Omega_s/(c_{snj} v_{zts})](v_{ds}/v_{zts} + c_j) 2\lambda_{Ts}\Gamma_n$
  $= \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [(v_{ds}/v_{zts} + c_j) c_j/\lambda_{Ts} + n\Omega_s b_{j0}(v_{ds}/v_{zts} + c_j)/(c_{snj} v_{zts})] 2\lambda_{Ts}\Gamma_n$
  $= \sum_\sigma r_{s\sigma}\omega_{ps}^2 b_j [(v_{ds}/v_{zts} + c_j) c_j/\lambda_{Ts} + n\Omega_s b_{j0}(1 - n\Omega_s/c_{snj})/(k_z v_{zts}^2)] 2\lambda_{Ts}\Gamma_n$.

B.3 Parallel propagation

For parallel propagation modes, $k_\perp = 0$, i.e., $b_s = 0$. We have $I_0 = 1$, $I_{n\neq 0} = 0$. $I_n(x) \simeq \frac{1}{n!}\left(\frac{x}{2}\right)^n$, when $x \ll n$ and $n \geq 0$. See also, [Gurnett2005 sec. 9.3]

Thus, we have

$$|\boldsymbol{D}(\omega,\boldsymbol{k})| = \begin{vmatrix} K_{xx} - \frac{c^2k^2}{\omega^2} & K_{xy} & 0 \\ K_{yx} & K_{yy} - \frac{c^2k^2}{\omega^2} & 0 \\ 0 & 0 & K_{zz} \end{vmatrix} = \left[\left(K_{xx} - \frac{c^2k^2}{\omega^2}\right)\left(K_{yy} - \frac{c^2k^2}{\omega^2}\right) - K_{xy}K_{yx}\right] K_{zz} = 0,$$

i.e., we obtain a electrostatic branch $K_{zz} = 0$, and two electromagnetic branch (noting that $K_{xx} = K_{yy}$ and $K_{xy} = -K_{yx}$)

$$D(\boldsymbol{k},\omega) = K_{xx} - \frac{c^2k^2}{\omega^2} \pm iK_{xy} = 0,$$

$$K_{xx} = 1 + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{\sigma=a,b} r_{s\sigma} \sum_{n=-\infty}^{\infty} \{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1 + \zeta_{sn}Z(\zeta_{sn})]\} n^2\Gamma_n/b_{s\sigma},$$

$$K_{xy} = \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{\sigma=a,b} r_{s\sigma} \sum_{n=-\infty}^{\infty} \{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1 + \zeta_{sn}Z(\zeta_{sn})]\} in\Gamma'_n.$$

Thus, the two electromagnetic branches are

$$\begin{aligned}
D(\boldsymbol{k},\omega) &= 1 - \frac{c^2k^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{\sigma=a,b} r_{s\sigma} \sum_{n=-\infty}^{\infty} \{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1 + \zeta_{sn}Z(\zeta_{sn})]\}[n^2\Gamma_n/b_{s\sigma} \mp n\Gamma'_n] \\
&= 1 - \frac{c^2k^2}{\omega^2} + \sum_s \frac{\omega_{ps}^2}{\omega^2} \sum_{\sigma=a,b} r_{s\sigma}\{\zeta_{s0}Z(\zeta_{sn}) + (\frac{1}{\lambda_{Ts\sigma}} - 1)[1 + \zeta_{sn}Z(\zeta_{sn})]\}|_{n=\pm1} \\
&= 0, \qquad (39)
\end{aligned}$$

with $\zeta_{sn} = \frac{\omega - k_z v_{ds} - n\Omega_s}{k_z v_{zts}}$, $\eta_{sn} = \frac{\omega - n\Omega_s}{k_z v_{zts}}$, $\omega_{sn} = \omega - k_z v_{ds} - n\Omega_s$. The above dispersion relation is very simple and can be solved similar to ES1D Landau damping case with the use of exact $Z(\zeta)$ function [Xie2013] (http://hsxie.me/codes/gpdf/).

Note: $\Gamma'_n(b) = (I'_n - I_n)e^{-b}$, $I'_n(b) = (I_{n+1} + I_{n-1})/2$, $I_{-n} = I_n$. For $b \to 0$, $I_0 = 1$, $I_{-1} = I_1 = \frac{b}{2}$, $I'_{-1} = I'_1 = (I_0 + I_2)/2 = 1/2$.

B.4 Electrostatic 3D

For electrostatic case, the dispersion relation reduces to Harris dispersion relation [Gurnett2005 sec.9.2]

$$D(\omega,\boldsymbol{k}) = 1 + \sum_s \frac{\omega_{ps}^2}{k^2} \sum_{n=-\infty}^{\infty} \int d\boldsymbol{v} \frac{J_n^2(k_\perp v_\perp/\Omega_s)}{\omega - k_\parallel v_\parallel - n\Omega_s} \Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp} + k_\parallel \frac{\partial f_{s0}}{\partial v_\parallel}\Big). \qquad (40)$$

And for loss cone distribution

$$
\begin{aligned}
D(\omega,\boldsymbol{k}) &= 1+\sum_s \frac{\omega_{ps}^2}{k^2}\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{J_n^2(k_\perp v_\perp/\Omega_s)}{\omega-k_\parallel v_\parallel-n\Omega_s}\Big(\frac{n\Omega_s}{v_\perp}\frac{\partial f_{s0}}{\partial v_\perp}+k_\parallel\frac{\partial f_{s0}}{\partial v_\parallel}\Big)\\
&= 1-\sum_s \frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\int d\boldsymbol{v}\frac{J_n^2(k_\perp v_\perp/\Omega_s)}{\omega-k_\parallel v_\parallel-n\Omega_s}2\Big[\frac{n\Omega_s}{v_{\perp ts\sigma}^2}+\frac{k_\parallel(v_\parallel-v_{ds})}{v_{zts}^2}\Big]f_{s0\sigma}\\
&= 1+\sum_s \frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b} r_{s\sigma}4\pi\sum_{n=-\infty}^{\infty}\int_0^\infty v_\perp dv_\perp J_n^2\Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{1}{k_z v_{zts}}Z(\zeta_{sn})+\frac{1}{v_{zts}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}f_{s0\perp\sigma}\\
&= 1+\sum_s \frac{\omega_{ps}^2}{k^2}\sum_{\sigma=a,b} r_{s\sigma}2\sum_{n=-\infty}^{\infty}\Big\{\frac{n\Omega_s}{v_{\perp ts\sigma}^2}\frac{1}{k_z v_{zts}}Z(\zeta_{sn})+\frac{1}{v_{zts}^2}[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\Gamma_n(b_{s\sigma})\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\Big\{\frac{\lambda_{Ts\sigma}n\Omega_s}{k_z v_{zts}}Z(\zeta_{sn})+[1+\zeta_{sn}Z(\zeta_{sn})]\Big\}\Gamma_n(b_{s\sigma})\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{1+\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\frac{\omega-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s}{k_z v_{zts}}Z(\zeta_{sn})\Gamma_n(b_{s\sigma})\Big\},
\end{aligned} \tag{41}
$$

After J-pole expansion,

$$
\begin{aligned}
D(\omega,\boldsymbol{k}) &\simeq 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{1+\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\frac{\omega-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s}{k_z v_{zts}}Z_J(\zeta_{sn})\Gamma_n(b_{s\sigma})\Big\}\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{1+\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\frac{\omega-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s}{k_z v_{zts}}\sum_{j=1}^{J}\frac{b_j}{\zeta_{sn}-c_j}\Gamma_n(b_{s\sigma})\Big\}\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{1+\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\Gamma_n(b_{s\sigma})[\omega-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s]\sum_{j=1}^{J}\frac{b_j}{\omega-c_{snj}}\Big\}\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{1+\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\Gamma_n(b_{s\sigma})\Big(-1+\sum_{j=1}^{J}\frac{b_j[c_{snj}-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s]}{\omega-c_{snj}}\Big)\Big\}\\
&= 1+\sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{\sum_{\sigma=a,b} r_{s\sigma}\sum_{n=-\infty}^{\infty}\Gamma_n(b_{s\sigma})\sum_{j=1}^{J}\frac{b_j[c_{snj}-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s]}{\omega-c_{snj}}\Big\}\\
&= 1+\sum_{snj}\frac{b_{snj}}{\omega-c_{snj}}=0,
\end{aligned} \tag{42}
$$

where we have defined $c_{snj}=k_z v_{ds}+n\Omega_s+k_z v_{zts}c_j$ and used $\sum_{j=1}^{J} b_j=-1$, and $b_{snj}=\sum_{\sigma=a,b}\frac{1}{k^2\lambda_{Ds}^2}r_{s\sigma}\Gamma_n(b_{s\sigma})b_j[c_{snj}-k_z v_{ds}-(1-\lambda_{Ts\sigma})n\Omega_s]=\sum_{\sigma=a,b}\frac{1}{k^2\lambda_{Ds}^2}r_{s\sigma}\Gamma_n(b_{s\sigma})b_j(k_z v_{zts}c_j+\lambda_{Ts\sigma}n\Omega_s)$.

The equivalent linear system can be

$$
\left\{\begin{array}{rcl}\omega n_{snj} &=& c_{snj}n_{snj}+b_{snj}E,\\ \omega E &=& -\sum_{snj}(c_{snj}n_{snj}+b_{snj}E).\end{array}\right. \tag{43}
$$

Note that we use $\omega E=...$, not directly $E=-\sum n_{snj}$ is to make the linear matrix $\boldsymbol{M}$ be sparse in $\omega\boldsymbol{X}=\boldsymbol{M}\cdot\boldsymbol{X}$, with $\boldsymbol{X}=[n_{snj},E]^T$.

If we further set $k_\perp = 0$, the ES3D case reduces to ES1D, i.e.,

$$\begin{aligned}
D(\omega, \boldsymbol{k}) &\simeq 1 + \sum_s \frac{1}{k^2\lambda_{Ds}^2}\Big\{ \sum_{\sigma=a,b} r_{s\sigma} \sum_{n=-\infty}^{\infty} \Gamma_n(b_{s\sigma}) \sum_{j=1}^{J} \frac{b_j[c_{snj} - k_z v_{ds} - (1-\lambda_{Ts\sigma})n\Omega_s]}{\omega - c_{snj}} \Big\} \\
&= 1 + \sum_s \frac{1}{k^2\lambda_{Ds}^2} \sum_{j=1}^{J} \frac{b_j(c_{s0j} - k_z v_{ds})}{\omega - c_{s0j}} \qquad (44) \\
&= 1 + \sum_{s0j} \frac{b_{s0j}}{\omega - c_{s0j}} = 0,
\end{aligned}$$

due to that $\Gamma_{n\neq 0}(0) = 0$ and $\Gamma_0(0) = 1$.

Last update: Saturday 27th October, 2018 16:33.